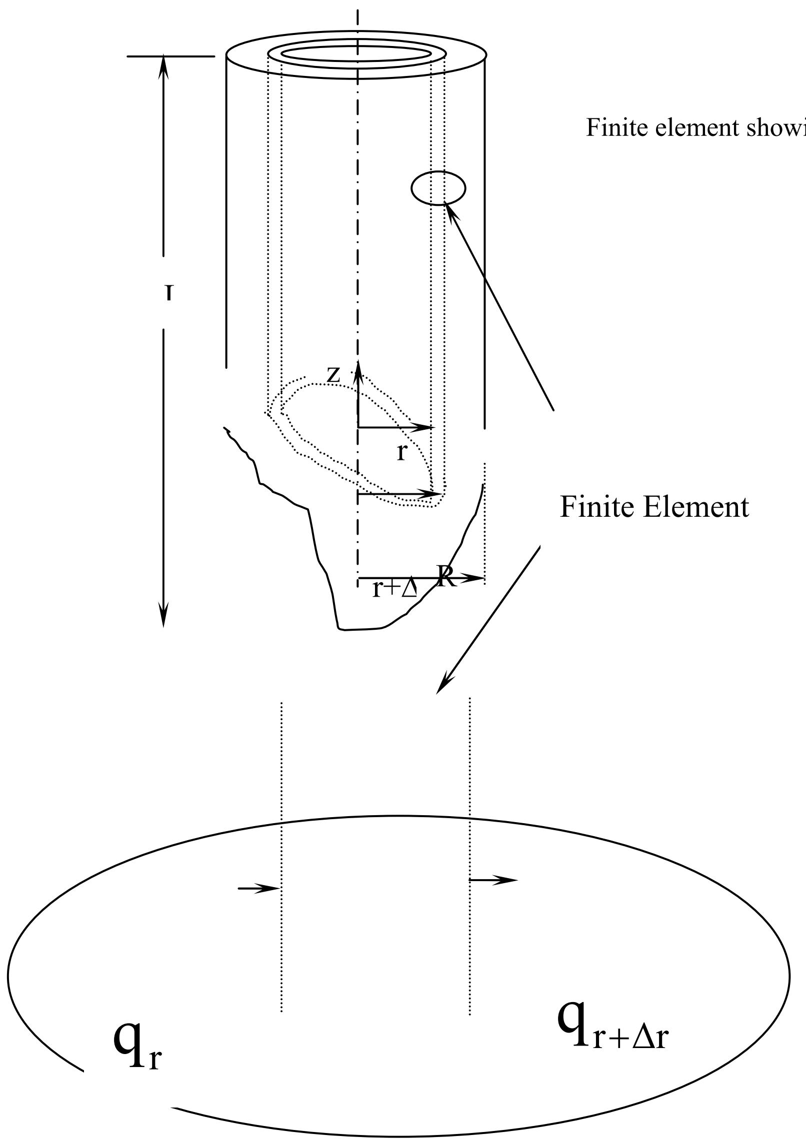
# Deriving Differential Equations

***Hour Exam 2001F***

1. Derive the Heat Equation for a one-dimensional, cylindrical coordinate, unsteady state heat conduction problem (i.e. – 1D USS HT   Cyl).

*As with all differential equation derivations, this one follows the same five steps:*

1. *Draw a sketch.*
2. *Establish an incremental element that has an incremental thickness in each direction of change.*




Finite element showing heat fluxes

3. *Perform a balance on the increment for the extensive quantity of interest (force, heat, etc.).*
4. *Divide through by all incremental terms ($\Delta X$, $\Delta y$, $\Delta t$, etc.) and take the limit as each goes to zero to obtain differential terms.*
5. *Substitute for any flux terms ($q$, $\tau$) the fundamental law that relates these fluxes to intensive property gradients.*

1 &2 The direction of change is r. Heat Balance

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc}$$

$$\left[2\pi L\left((rq)_{\text{r}} - (rq)_{r+\Delta r}\right) + 0\right]\Delta t = 2\pi r L \Delta r \rho Cp (T_{t+\Delta t} - T_t)$$

4. Divide by $2\pi L\Delta r\Delta t$

$$\frac{(rq)_{\text{r}} - (rq)_{r+\Delta r}}{\Delta r} = r\rho Cp \frac{(T_{t+\Delta t} - T_t)}{\Delta t}$$

In the limit as each delta term approaches 0

$$-\frac{\partial r q_{\text{r}}}{\partial r} = r\rho Cp \frac{\partial T}{\partial t}$$

5. Substitute Fourier's Law of Heat Conduction: $q_r = -k\dfrac{\partial T}{\partial r}$

$$\alpha \frac{\partial \ (rq_{\text{r}})}{\partial r} = r\frac{\partial T}{\partial t}$$

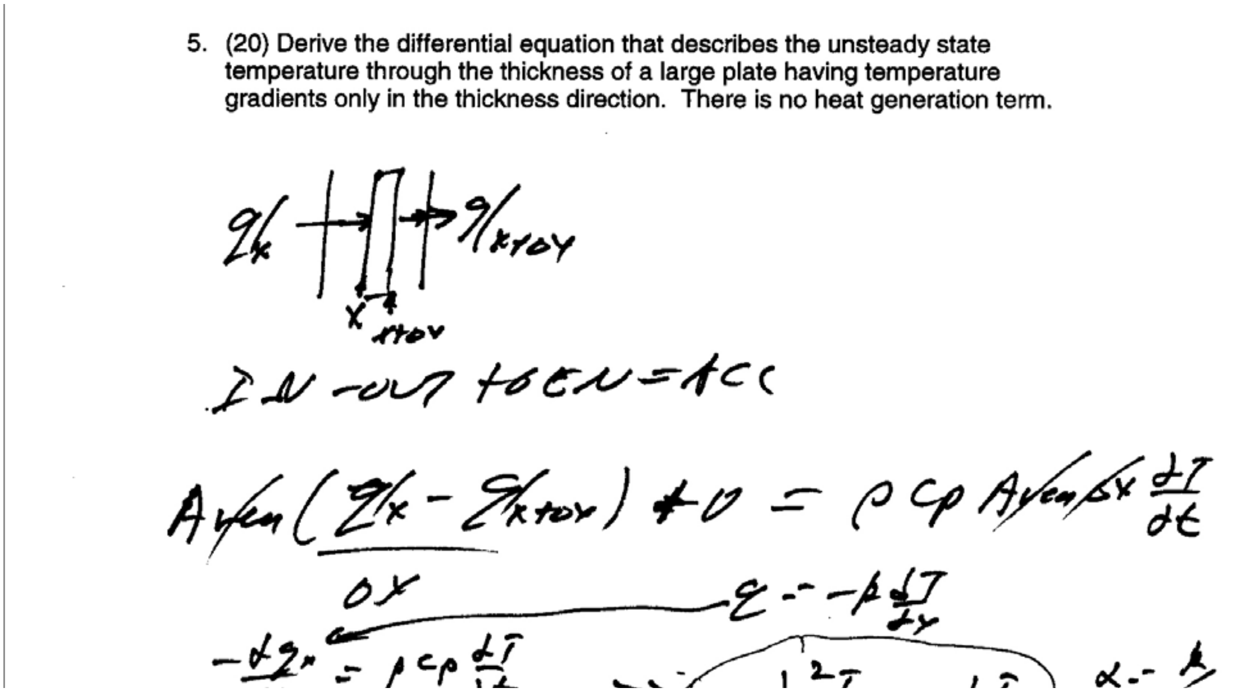
Expanding the first term and dividing by r gives

$$\alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] = \frac{\partial T}{\partial t}$$

where $\alpha = \dfrac{k}{\rho Cp}$

***Hour Exam 1998F***

5. (20) Derive the differential equation that describes the unsteady state temperature through the thickness of a large plate having temperature gradients only in the thickness direction. There is no heat generation term.



5. (20) Derive the differential equation that describes the unsteady state temperature through the thickness of a large plate having temperature gradients only in the thickness direction. There is no heat generation term.

$q/_x \rightarrow \quad \rightarrow q/_{x+\Delta x}$

IN − OUT + GEN = ACC

$$A_{yz}\left(\frac{q/_x - q/_{x+\Delta x}}{\Delta x}\right) + 0 = \rho\, C_p\, A_{yz}\, \Delta x \frac{dT}{dt}$$

$$q = -k\frac{dT}{dx}$$

$$-\frac{dq_x}{dx} = \rho\, c_p \frac{dT}{dt}$$