8 – Interpolation

Every engineering and science student has performed linear interpolation to obtain an estimate of a dependent value occurring between two corresponding independent values listed in a table. The term *estimate* is used because tabulated values rarely follow a strictly linear behavior. Otherwise, the table would have not been generated. A simple linear equation would have been used to describe the data. Since the data in tables is usually non-linear, the use of more than two adjacent tabular values might be used to determine the curvature (quadratic) or higher order variations of the data.

If N tabular values are used in a polynomial to approximate the value of a function for which regular values of the function are known, an approximation of the data will be of the order N-1. If it happens to be that tabular data derives from a polynomial of order N, then N+1 data points should allow the determination of the exact coefficients in the original polynomial. This feature is demonstrated in this chapter. However, beyond this bit of cleverness determining the original function, interpolation has many uses in engineering practice. First, it allows compaction of data in databases (sometimes embedded on a computer chip) that can be retrieved to considerable great accuracy using high order interpolation. Secondly, it provides a means of approximating generic functions needed in derivations. The error in such approximations is determinable since such approximations observe the Mean Value Theorem of Derivatives. This is employed in the derivation of Simpson's Rules (§Numerical Integration).

8.1 – Elementary Interpolation

Linear: Linear interpolation of a function of x is described by the expression

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0} \tag{8.1}$$

where the value of f(x) is sought given $f(x_0)$ and $f(x_1)$.

Solving Eq. (8.1) gives

$$f(x)=f(x_0)+\frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) \tag{8.2}$$

If one considers Eq. (8.2), it has the general form

$$\mathrm{f(x)=b_0+b_1(x-x_0)} \tag{8.3}$$

where

$$b_0 = f(x_0) \tag{8.4}$$

$$b_1=\frac{f(x_1)-f(x_0)}{x_1-x_0} \tag{8.5}$$

This leads to the following form for approximating a function with a polynomial:

$$f(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)+b_3(x-x_0)(x-x_1)(x-x_2)+... \tag{8.6}$$

Equation (8.6) is called *Newton's Difference Equation*.

Quadratic: The straight-line interpolation above may be improved by using higher order approximations. For example, a quadratic interpolation using three values of the function would account for the curvature in the function. Four points would account for the rate of change of curvature, etc. Eq. (8.6) is a very convenient starting point for performing such interpolations since the *b* terms are related to the known values of the function at known values of x.

For quadratic interpolation the third term in Eq. (8.6) is needed. The value of b_2 may be determined by setting x = x_2 and solving as follows:

$$(x_2 - x_0)b_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_1)} - \frac{[f(x_1) - f(x_0)](x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)} \tag{8.7}$$

$$(x_2 - x_0)b_2 = \frac{[f(x_2) - f(x_0)](x_1 - x_0) - [f(x_1) - f(x_0)](x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)} \tag{8.8}$$

If the term $f(x_1)(x_2 - x_1)$ is added to the first term in the numerator and subtracted from the second term, the resulting terms may be combined as shown at the end of this chapter to give

$$b_2 = \frac{\dfrac{[f(x_2) - f(x_1)]}{(x_2 - x_1)} - \dfrac{[f(x_1) - f(x_0)]}{(x_1 - x_0)}}{(x_2 - x_0)} \tag{8.9}$$

The pattern of the b coefficients is now established and can be determined easily using a divided difference table as described in the next section.

<u>8.2 – Newton's Divided Difference Table</u>

The progression of the b terms is now seen and may be written for any order interpolation. For convenience a *Divided Difference Table*, as shown in Examples 8a and 8b, is frequently used to keep track of the terms needed. The proper *divided difference* notation is

$$\begin{aligned} f(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) + \ldots \end{aligned} \tag{8.10}$$

The values appearing in Table 1 will be used to illustrate the use of a divided difference table. The first column contains the regularly-spaced values of x corresponding to the function values f(x) in the second column. The third and subsequent columns contain values of b_1, b_2, - - b_N labeled in accordance with the notation in Eq. (8.10) and computed from Eqs. (8.4), (8.5), (8.9), and the continuing series of computations.

Example 8a

Estimate the value of f(1.21) from the data in Table 1 using a third order approximation (cubic).

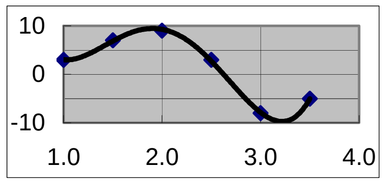
$$f(1.21) = 3 + 8.0*(1.21-1) - 4.0*(1.21-1)(1.21-1.5) - 8.0*(1.21-1)(1.21-1.5)(1.21-2) = 4.539 \tag{8.11}$$

The error in this approximation depends on several things: 1) the number of significant figures in the divided difference table values used in the interpolation and 2) the error in the approximation. From the Taylor Series and the Mean value Theorem this would be on the order of

$$\text{Error} = \frac{f''''(\xi)h^4}{4!} = O(0.5)^4 = O(0.0125) \tag{8.12}$$

Table 1 – Sample Interpolation Data with Corresponding Divided Difference Coefficients.

	0th order	1st order	2nd order	3rd order	4th order	etc
x	**f(x)**	**$f[x_0,x_1]$**	**$f[x_0,x_1,x_2]$**	**$f[x_0,x_1,x_2,x_3]$**	**$f[x_0,x_1,x_2,x_3,x_4]$**	
1.0	3					
		8.00				
1.5	7		-4.0000			
		4.00		-8.00000		
2.0	9		-16.0000		6.00000	
		-12.00		4.00000		1.866667
2.5	3		-10.0000		10.66667	
		-22.00		25.33333		
3.0	-8		28.0000			
		6.00				
3.5	-5					
x	**1.21**					
	0th order Apprx	1st order Apprx	2nd order Apprx	3rd order Apprx	4th order Apprx	5th order Apprx
f(1.21)	3.000	4.680	4.924	4.539	4.166	4.106

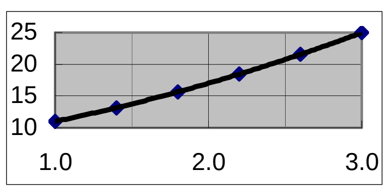


Example 8b

The divided difference values in Table 2 are from a polynomial. Determine the order and coefficients of the polynomial.

Table 2 –Divided Difference Values from a Polynomial.

	0th order	1st order	2nd order	3rd order
x	**f(x)**	**$f[x_0,x_1]$**	**$f[x_0,x_1,x_2]$**	**$f[x_0,x_1,x_2,x_3]$**
1.0	11			
		5.40		
1.4	13.16		1.0000	
		6.20		0.00000
1.8	15.64		1.0000	
		7.00		0.00000
2.2	18.44		1.0000	
		7.80		0.00000
2.6	21.56		1.0000	
		8.60		
3.0	25			



Since the third and higher divided differences are zero, the order of the polynomial is 2. The coefficients of the original equation are determined as follows:

$$f(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$f(x) = 11 + 5.40(x - 1.0) + 1.000(x - 1.0)(x - 1.4) \qquad (8.13)$$

$$f(x) = x^2 + 3x + 7$$

Summary of the Divided Difference Table Interpolation Method

$$\begin{aligned} f(x) = \; & f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ & + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) + \ldots \end{aligned} \qquad (8.14)$$

where

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$f[x_0, x_1, x_2, x_3] = \frac{\dfrac{\dfrac{f(x_3) - f(x_2)}{x_3 - x_2} - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1} - \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}}{x_3 - x_0}$$

..

..

8.3 – Newton's Difference Table

The divided difference table is a means of organizing computations to be used for use in Newton's interpolation equation shown in Eq. (8.6). An alternate method of organizing these computations is to construct the simpler *Newton's Difference Table*. The values in the difference table are not divide the differences of the independent values. Of course, then, the equations used to perform interpolations must be divided by what would have been the missing accumulated differences in the independent values. In the linear term that denominator is as shown in Eq. (8.5) is

$$(x_1 - x_0) = h \qquad (8.15)$$

The denominator for the quadratic term is easily found to be $2h^2$ by manipulating Eq. (8.9) as follows:

$$b_2 = \frac{\dfrac{\left(f(x_2)-f(x_1)\right)}{\left(x_2-x_1\right)}-\dfrac{\left(f(x_1)-f(x_0)\right)}{\left(x_1-x_0\right)}}{\left(x_2-x_0\right)} = \frac{\dfrac{\left(f(x_2)-f(x_1)\right)}{h}-\dfrac{\left(f(x_1)-f(x_0)\right)}{h}}{2h}$$

$$= \frac{\left(f(x_2)-f(x_1)\right)-\left(f(x_1)-f(x_0)\right)}{2h^2} \tag{8.16}$$

Summary of the Difference Table Interpolation Method

$$f(x)=f(x_0)+\frac{\Delta f(x_0)}{h}(x-x_0)+\frac{\Delta^2 f(x_0)}{2!h^2}(x-x_0)(x-x_1)+\frac{\Delta^3 f(x_0)}{3!h^3}(x-x_0)(x-x_1)(x-x_2)+\ldots \tag{8.17}$$

where $h = x_{i+1}-x_i$

$$\Delta f(x_0)=f(x_1)-f(x_0)$$

$$\Delta^2 f(x_0)=\left[f(x_2)-f(x_1)\right]-\left[f(x_1)-f(x_0)\right]$$

$$\Delta^3 f(x_0)=\left\{\left[f(x_3)-f(x_2)\right]-\left[f(x_2)-f(x_1)\right]\right\}-\left\{\left[f(x_2)-f(x_1)\right]-\left[f(x_1)-f(x_0)\right]\right\}$$

....

Example 8c

Rework Example 8b using a difference table.

The coefficients of the original equation are determined as follows using Eq. (8.17):

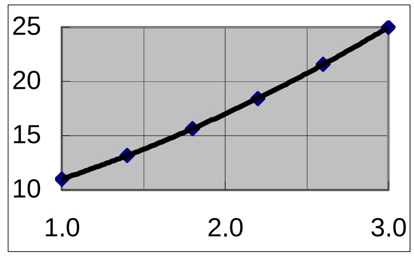
$$f(x)=f(x_0)+\frac{\Delta f(x_0)}{h}(x-x_0)+\frac{\Delta^2 f(x_0)}{2!h^2}(x-x_0)(x-x_1)$$

$$f(x)=11+\frac{2.16(x-1.0)}{h}+\frac{0.3200(x-1.0)(x-1.4)}{2!h^2} \qquad h=0.4 \tag{8.18}$$

$$f(x)=x^2+3x+7$$

Table 3 –Difference Table for the Function in Table 2.

	0th order	1st order	2nd order	3rd order	4th order	etc
x	**f(x)**	**$\Delta f[x]$**	**$\Delta^2 f[x]$**	**$\Delta^3 [x]$**	**$\Delta^4 f[x]$**	
1.0	11					
		2.16				
1.4	13.16		0.3200			
		2.48		0.00000		
1.8	15.64		0.3200			
		2.80		0.00000		
2.2	18.44		0.3200			
		3.12		0.00000		
2.6	21.56		0.3200			
		3.44				
3.0	25					



Example 8d

Rework Example 8a using a Difference Table.

$$f(x) = f(x_0) + \frac{\Delta f(x_0)}{h}(x - x_0) + \frac{\Delta^2 f(x_0)}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 f(x_0)}{3!h^3}(x - x_0)(x - x_1)(x - x_2)$$

$$= 3 + \frac{4}{h}(1.21 - 1) - \frac{2}{2!h^2}(1.21 - 1)(1.21 - 1.5) - \frac{6}{3!h^3}(1.21 - 1)(1.21 - 1.5)(1.21 - 2.0)$$

$$= 4.539$$

where $h = 0.5$

Table 3 –Difference Table for the Data in Table 1.

	0th order	1st order	2nd order	3rd order	4th order	etc
x	**f(x)**	**$\Delta f[x]$**	**$\Delta^2 f[x]$**	**$\Delta^3 [x]$**	**$\Delta^4 f[x]$**	
1.0	3					
		4.00				
1.5	7		-2.0000			
		2.00		-6.00000		
2.0	9		-8.0000		9.00000	
		-6.00		3.00000		7.00000
2.5	3		-5.0000		16.00000	
		-11.00		19.00000		
3.0	-8		14.0000			
		3.00				
3.5	-5					

8.4 – Lagrangian Interpolation

Lagrangian interpolation is more easily applied to automated computer routines. The interpolation formulation is

$$f(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

$$\text{where} \quad L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} \tag{8.19}$$

Example 8e

Rework Example 8a using Lagrangian Interpolation.

Table 4 –Data from Example 8a

x	f(x)
1.000	3.000
1.500	7.000
2.000	9.000
2.500	3.000
3.000	-8.000
3.500	-5.000

The result is as follows:

$$f(1.21) \quad = 3 * L_0(x) + 7 * L_1(x) + 9 * L_2(x) + 3 * L_3(x)$$

$$L_0(x) = \frac{(1.21 - 1.5)(1.21 - 2.0)(1.21 - 2.5)}{(1.0 - 1.5)(1.0 - 2.0)(1.0 - 2.5)} = 0.3940$$

$$L_1(x) = \frac{(1.21 - 1.0)(1.21 - 2.0)(1.21 - 2.5)}{(1.5 - 1.0)(1.5 - 2.0)(1.5 - 2.5)} = 0.8560$$

$$L_2(x) = \frac{(1.21 - 1.0)(1.21 - 1.5)(1.21 - 2.5)}{(2.0 - 1.0)(2.0 - 1.5)(2.0 - 2.5)} = \text{-}0.3142 \tag{8.20}$$

$$L_3(x) = \frac{(1.21 - 1.0)(1.21 - 1.5)(1.21 - 2.0)}{(2.5 - 1.0)(2.5 - 1.5)(2.5 - 2.0)} = 0.0641$$

$$f(1.21) \quad = 4.539$$

Appendix A: Derivation of b_2 in Eq. (8.6)

$$\left(x_2 - x_0\right) b_2 = \frac{\left[f(x_2) - f(x_0)\right]\left(x_1 - x_0\right) - \left[f(x_1) - f(x_0)\right]\left(x_2 - x_0\right)}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{\left[f(x_2) - f(x_0)\right]\left(x_1 - x_0\right) + \mathbf{f(x_1)\left(x_2 - x_1\right)} - \left[f(x_1) - f(x_0)\right]\left(x_2 - x_0\right) - \mathbf{f(x_1)\left(x_2 - x_1\right)}}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{f(x_2)\left(x_1 - x_0\right) - f(x_0)\left(x_1 - x_0\right) + f(x_1)\left(x_2 - x_1\right) - f(x_1)\left(x_2 - x_0\right) + f(x_0)\left(x_2 - x_0\right) - f(x_1)\left(x_2 - x_1\right)}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{f(x_2)\left(x_1 - x_0\right) - \left[f(x_1)\left(x_2 - x_0\right) - f(x_1)\left(x_2 - x_1\right)\right] - f(x_1)\left(x_2 - x_1\right) - \left[f(x_0)\left(x_1 - x_0\right) - f(x_0)\left(x_2 - x_0\right)\right]}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{f(x_2)\left(x_1 - x_0\right) - \left[f(x_1)\left(x_1 - x_0\right)\right] - f(x_1)\left(x_2 - x_1\right) - \left[f(x_0)\left(x_1 - x_2\right)\right]}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{f(x_2)\left(x_1 - x_0\right) - \left[f(x_1)\left(x_1 - x_0\right)\right]}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)} - \frac{f(x_1)\left(x_2 - x_1\right) - \left[f(x_0)\left(x_2 - x_1\right)\right]}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$= \frac{\left[f(x_2) - f(x_1)\right]\left(x_1 - x_0\right)}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)} - \frac{\left[f(x_1) - f(x_0)\right]\left(x_2 - x_1\right)}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)}$$

$$b_2 = \left[\frac{\dfrac{\left[f(x_2) - f(x_1)\right]}{\left(x_2 - x_1\right)} - \dfrac{\left[f(x_1) - f(x_0)\right]}{\left(x_1 - x_0\right)}}{\left(x_2 - x_0\right)}\right]$$

