Root Finding Methods

Review

Root Finding

- The root finding process involves finding a root, or solution, of an equation of the form $f(x) = 0$.
- Therefore, the first step for all root finding problems is to rearrange the equation so that all the terms appear on the left side.

Why Numerical Methods?

- Exact Solutions

  For some functions, we can calculate roots exactly; e.g.,

  - Polynomials up to degree 4
  - Simple transcendental functions, such as sin x= 0

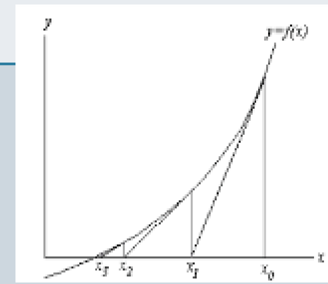
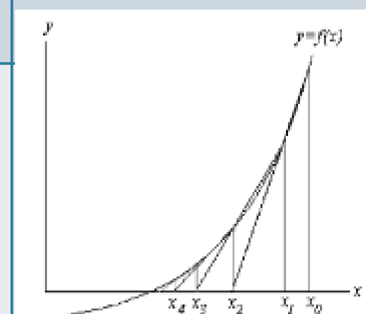
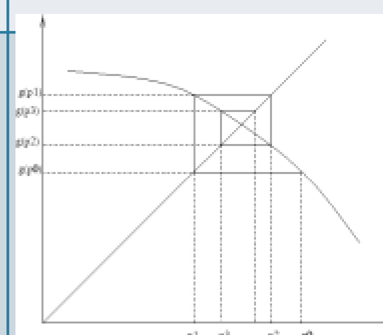
- Numerical Methods

  Give approximation but accurate solutions to hard problems that can't be solved directly.

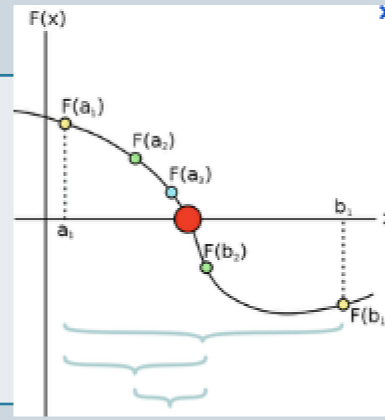
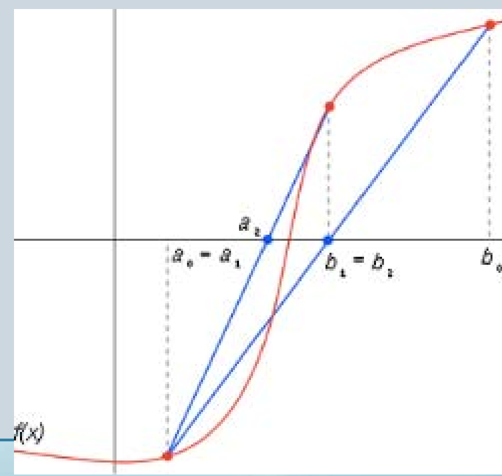
Non-linear Equations w/One Unknown

- Bracketing Methods
  - Trial and error
  - Bisection
  - False position
- Non-Bracketing Methods
  - Newton's Method
  - Secant Method
  - One-point iteration

Non-Bracketing Methods

Methods	Formula	Initial Guess	Illustration	Note
Newton's	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	x_0		If no convergence, choose a new initial guess
Secant	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	x_0, x_1		If no convergence, choose a new initial guess
One-point iteration	$x_{n+1} = g(x_n)$			To converge, $\lvert g'(x)\rvert<1$

Bracketing Methods

Methods	Formula	Initial Guess	Illustration	Note
Trial and error		x_0, x_1		Divide previous interval by 10 for each iteration.
Bisection	$x_n = \dfrac{x_{Ln} + x_{Rn}}{2}$	x_L, x_R		If $f(x_L)f(x_n)<0$, set x_n as new x_R If $f(x_R)f(x_n)<0$, set x_n as new x_L
False Position	$x_n = \dfrac{x_{Ln} f(x_{Rn}) - x_{Rn} f(x_{Ln})}{f(x_{Rn}) - f(x_{Ln})}$	x_L, x_R		If $f(x_L)f(x_n)<0$, set x_n as new x_R If $f(x_R)f(x_n)<0$, set x_n as new x_L

Non-linear Equation Systems

- Accurate solution:
  - Gaussian elimination: reduces the coefficient matrix to triangular form
  - Gaussian-Jordan elimination: reduces the coefficient matrix further to diagonal form
- Numerical methods (approximation)
  - Gauss-Seidel: Arrange equations to diagonally dominant matrix: $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$
  - Newton-Raphson

- Example: solve

$$x + 3y + 2z = 11$$
$$2x + y = 6$$
$$-x + 2y - 2z = 9$$

➢ Gauss:

Coefficient matrix:

x	y	z	RHS
1	3	2	11
2	1	0	6
-1	2	-2	9

Final matrix:

x	y	z	RHS
1	0	-2/5	7/5
0	1	4/5	16/5
0	0	-4	4

Reduce further to have a diagonal matrix:

➢ Gauss- Jordan

x	y	z	RHS
1	0	0	1
0	1	0	4
0	0	1	-1

Gauss-Seidel Method

Solve the following linear system using Gauss-Seidel method

$$2x-6y-4z = -2$$

$$5x+2y-3z = 14$$

$$3x+3y+8z = 42$$

Rearrange the equations to have a diagonally dominant system:

$5x+2y-3z = 14 \rightarrow x= (14-2y+3z)/5$

$2x-6y-4z = -2 \rightarrow y=(-2-2x+4z)/(-6)$

$3x+3y+8z = 42 \rightarrow z=(42-3x-3y)/8$

Select initial guesses for y and z, run iterations using the equation above.

Note: The latest value for each variable is always used.

Newton-Raphson Method

- A Newton-Raphson method for solving the system of linear equations requires the evaluation of a matrix, known as the Jacobian of the system, which is defined as:

$$J = \frac{\partial(f_1, f_2, \dots f_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \dots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \dots & \partial f_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix} = \left[\frac{\partial f_i}{\partial x_j}\right]_{n\times n}$$

- If $\mathbf{x} = \mathbf{x}_0$ (a vector) represents the first guess for the solution, successive approximations to the solution are obtained from

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}^{-1} \cdot \mathbf{f}(\mathbf{x}_n)$$