

7 Root Finding

Procedures used to solve equations can be divided into two classifications: analytical and numerical. Analytical methods involve the explicit solution of equations in terms of the unknowns and constants in the equations. For example, the solution to the quadratic equation $ax^2 + bx + c$ is $(-b \pm \sqrt{b^2 - 4ac})/(2a)$. This analytical solution is explicit and was obtained by the manipulation of the quadratic equation in terms of the unknown and constants. On the other hand, numerical methods arrive at solutions through the manipulation of numbers. This manipulation is usually in the form of successive guesses of the solution. In these cases, numerical methods are designed so that each guess yields a result, which is closer to the root than the guess. By using this result for the next guess, one may converge on the root. The objective of this chapter is to present useful numerical methods that can be used to solve non-linear equations with one independent variable and to solve first-order simultaneous equations. The importance of numerical methods has increased tremendously in recent years because of the widespread availability of digital computers.

7.1 Non-linear Equations in one Unknown

Linear equations in one unknown are equations in which the unknown is raised to the first power only. Also, logarithmic or trigonometric functions involving the unknown do not appear. Therefore, linear equations in one unknown can be rearranged and solved easily. Examples of linear equations in one unknown are

$$x = 3x + 2 \tag{7.1}$$

$$10y = 6 \tag{7.2}$$

Their solution is so trivial that they are ignored in the subject of numerical methods. On the other hand, non-linear equations with one unknown are frequently difficult to solve analytically and in some cases can be solved only by numerical methods. Examples of non-linear equations are

$$x^2 + x = 1$$

$$x^5 = 2.6$$

$$\log x + x = 4 \tag{7.3}$$

$$\tan x = 4 + \cos x$$

$$x^{-1} + x^4 + 1 = x \log x - 6 \tan x$$

$$\log x - \left(\frac{\tan x}{(1 + x^{0.21})} \right) = 0$$

All numerical methods for solving non-linear equations begin with the equation in the form

$$f(x) = 0 \tag{7.4}$$

This is achieved by rearranging the equation so that all of the terms appear on the left side. The objective of numerical methods for solving all non-linear equations in one unknown is then the same. It is to find

the values of x which make the function of x equal to zero. When more than one root exists for the equation, the value of the function will equal zero at more than one value of x . This is illustrated in Figure 1.

Unlike rigorous analytical solutions which result in exact values of the root, numerical methods result in an approximation of the root. The accuracy of the approximation may be set at any desired level, but should be established by the accuracy needed in the solution. This depends upon the nature and use of the mathematical model being solved. Since numerical methods consist of repetitive approximations of the root which are increasingly accurate near the root, the repetitive procedure is terminated when the difference in successive approximations is less than the established error criterion.

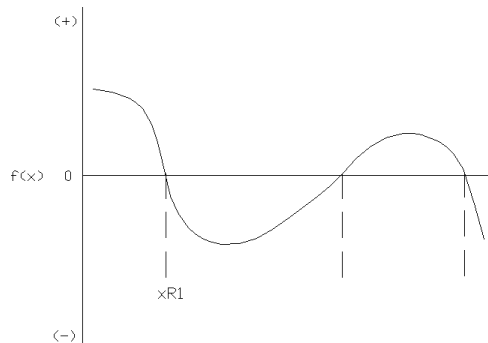


Figure 1-Behavior of a Typical Non-linear Function.

Trial and Error: The most easily understood method of finding a root, x_R , is by the trial and error method. The method consists of evaluating $f(x)$ at the regular values of x_1, x_2, x_3, \dots as shown in Figure 2, until the value of $f(x)$ changes sign. In Figure 2 this occurs between x_2 and x_3 . The interval between x_2 and x_3 is then divided into ten equal intervals and the procedure is repeated. The accuracy in the value of x_R determined by this method depends on the size of the last increment. Normally, a difference of 10^{-6} would be considered acceptable. However, if $x_R = 2 \times 10^{-13}$ that error criterion would be much too large. On the other hand, if $x_R = 4.1 \times 10^{42}$, such accuracy would most likely be ridiculously stringent. Therefore, dividing by the value of x near the root usually normalizes the error criterion. Thus, a common error criterion is $\left| \frac{x_1 - x_2}{x_1} \right| \leq 10^{-6}$ where x_1 and x_2 are the values of x which give values of $f(x)$ of opposite sign. That is, the accuracy in x_R is one-millionth the value of x_R .

The trial and error method is very reliable in that it nearly always finds a root. However, it is not used extensively because it is very inefficient and requires more computer time than the following methods.

Interval Halving (Bisection) The interval halving method is the same as the trial and error method except that the interval is divided into two rather than ten increments. This change yields higher efficiency, but is still not much better than the trial and error method. The interval halving method is used to solve the problem in Example 7a.

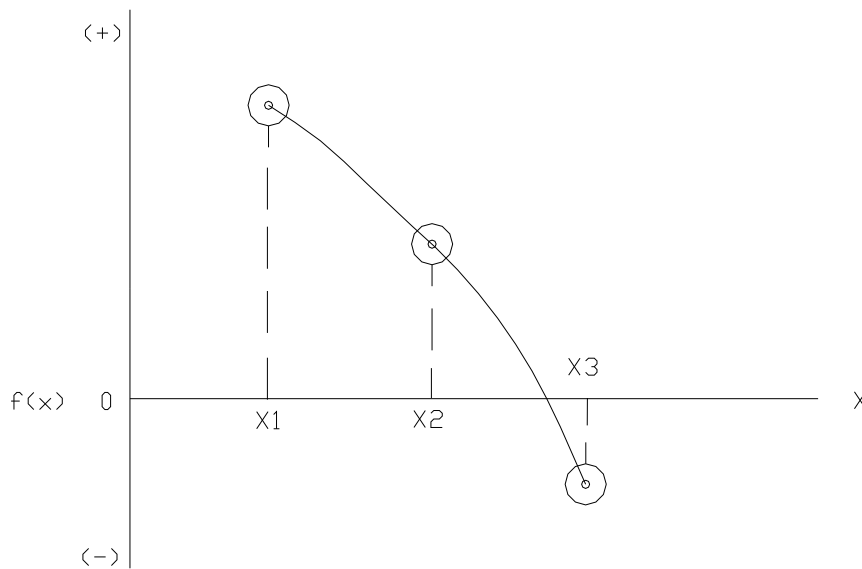


Figure 2-Trial and Error Method.

Secant Method Figure 3 illustrates the secant method. The first step of the method is to evaluate $f(x)$ at two arbitrarily selected values of x denoted in Figure 3a as x_1 and x_2 . A straight line through the two points having the coordinates $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is used to determine an estimate of the root. The estimate is the value of x where the straight line intersects the x -axis. This estimate is denoted as x_3 . If x_3 is nearly the same as x_2 , the function $f(x)$ is very steep near x_2 and the root is very close to x_3 . However, it is more likely that x_3 will not be close to x_2 . In that case, the procedure is repeated using the coordinates $(x_2, f(x_2))$ and $(x_3, f(x_3))$ as shown in Figure 3b to obtain a new estimate x_4 . This procedure is repeated until the change in x becomes acceptably small. As before, a reasonable error criterion is $|(x_{n+1} - x_n)/x_n| \leq 10^{-6}$.

Example 7a

The mathematical model for the rate of heat loss in BTU's per minute for a hot steel ingot is

$$Q = 4.53T + 1.14 \cdot 10^{-8}T^4 - 3417 \quad (7.5)$$

where T is in degrees Fahrenheit. What is the temperature of the ingot when the heat loss rate is 12,000 BTU's per minute? The required accuracy is one Fahrenheit degree.

Substitution of 12,000 for Q and rearranging gives

$$f(T) = 4.53T + 1.14 \cdot 10^{-8}T^4 - 15,417 = 0$$

Solving by the interval halving method with an initial value of zero for T and an initial step size of 512 gives the following results. The step size was selected to avoid fractional temperatures. That is, it can be halved without incurring fractions. Such a consideration is not a necessary part of the interval halving method.

<u>T</u>	<u>f(T)</u>	<u>Remarks</u>
0	-15,417	-
512	-12,314	-
1024	+ 1,756	$512 < T_R < 1024$
768	-7,972	$768 < T_R < 1024$
896	-4,011	$896 < T_R < 1024$
960	-1,386	$960 < T_R < 1024$
992	+ 116	$960 < T_R < 992$
976	- 651	$976 < T_R < 992$
984	- 272	$984 < T_R < 992$
988	- 79	$988 < T_R < 992$
990	+ 18	$988 < T_R < 990$
		$T_R = 989 \pm 1$

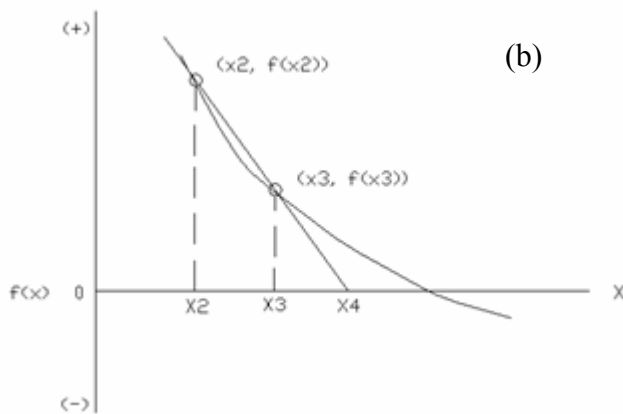
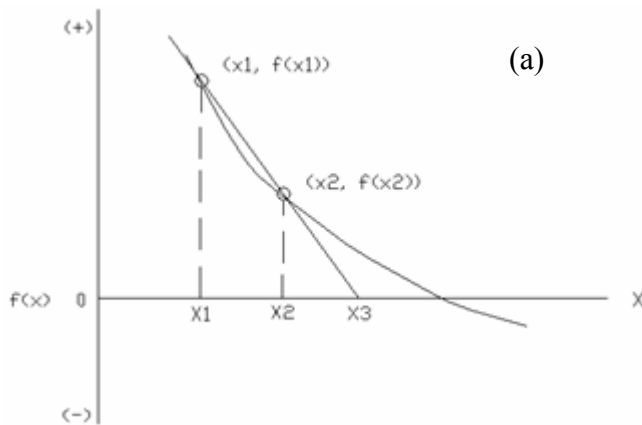


Figure 3 Secant Method: a) First Approximation
b) Second Approximation

The general equation for the new estimate of the root, x_{n+1} , is derived using Figure 4. Since the small, cross-hatched triangle is similar to the large speckled triangle,

$$\frac{x_{n+1} - x_n}{-f(x_n)} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad (7.6)$$

Rearranging gives

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad (7.7)$$

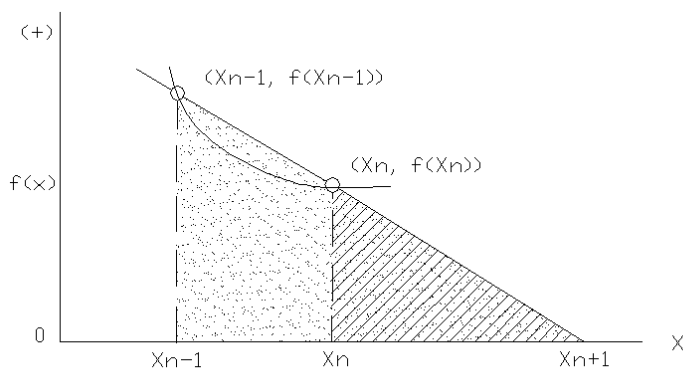


Figure 4-Derivation of the Secant Method Equation.

The secant method will normally converge on the root very rapidly. However, under some circumstances the method may fail to locate a root. For example, if the two arbitrarily selected points x_1 and x_2 define a straight line of no slope the first estimate of the root will be undefined. In such a case, two new starting points must be used.

Example 7b

Solve the problem in Example 7a by the secant method. As before

$$f(T) = 4.53T + 1.14 \times 10^{-8} T^4 - 15,417 = 0 \quad (7.8)$$

If the initial guesses are $T_1 = 500$ and $T_2 = 600$, the approximation resulting from the use of Eq. (7.2) is

$$T_{n+1} = T_n - \frac{f(T_n)(T_n - T_{n-1})}{f(T_n) - f(T_{n-1})} \quad (7.9)$$

$$T_{n+1} = 600 - \frac{(-11,222)(600 - 500)}{(-11,222) - (-12,439)} T_3 = 1622.1^{\circ} \text{F} \quad (7.10)$$

The large difference between T_2 and T_3 makes it clear that additional iterations are required. Using T_2 and T_3 , in Eq. (7.7) for T_{n-1} and T_n gives

$$T_4 = 762.14^{\circ} \text{F} \quad (7.11)$$

Continuing this procedure until the difference between the approximations is less than one $^{\circ}\text{F}$ gives

$$T_5 = 863.74^{\circ} \text{F}$$

$$T_6 = 1040.88^{\circ} \text{F}$$

$$T_7 = 980.32^{\circ} \text{F}$$

$$T_8 = 988.98^{\circ} \text{F}$$

$$T_9 = 989.63^{\circ} \text{F}$$

Therefore, the answer is 989.62°F .

Newton's Method: Since Newton is the Father of Calculus, one might correctly expect that Newton's method involves the use of calculus. Specifically, calculus is used to determine the slope of a tangent to the function $f(x)$ at some estimate of the root, x_n . As shown in Figure 5, the intersection of this tangent with the x-axis yields a new approximation of the root, x_n .

The equation used to calculate x_{n+1} is easily derived by equating the first derivative of the function of x_n to the slope of the tangent.

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \quad (7.12)$$

Solving for x_{n+1} gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (7.13)$$

This equation is used repetitively until the difference between x_n and x_{n+1} is small enough to satisfy a predetermined error criterion.

Like the secant method, Newton's method is more efficient than the trial and error method and the interval-halving method. Additionally, it is usually more efficient than the secant method, because it requires fewer calculations per approximation. However, Newton's method can oscillate about a minimum or maximum which is isolated from the root. Figure 6 illustrates such oscillation. Functions like the one shown in Figure 6 can also lead to oscillation when using the secant method. In some cases, the shape of the function is such that successive approximations diverge. That is, they oscillate about some position, but the distance of the approximations from this position continually increase. Selecting new starting approximations of the root can usually eliminate oscillation and divergence.

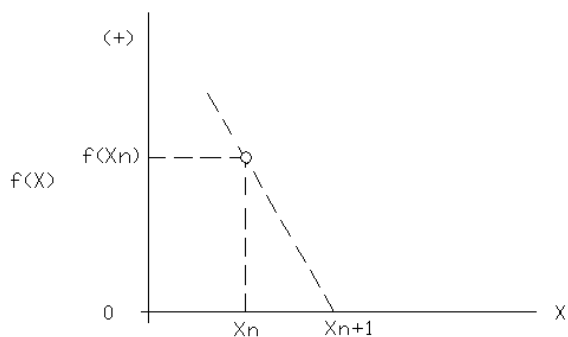


Figure 7.5-Newton's Method

Example 7c

Find the square root of ten by Newton's method. The answer must be accurate to within four decimal places.

If the square root is x , then

$$x^2 = 10. \quad (7.14)$$

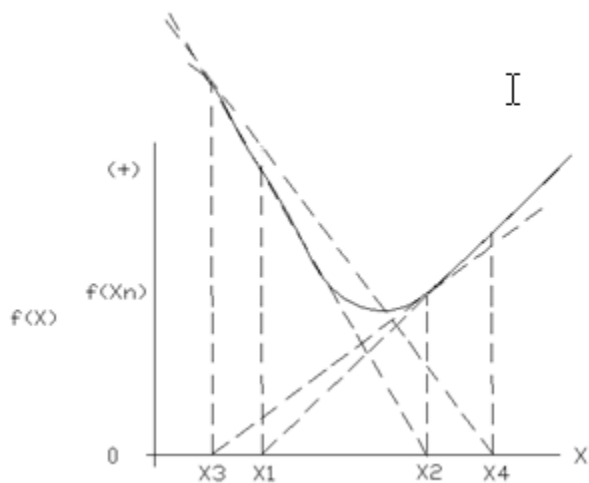


Figure 6-Newton's Method Oscillation

Therefore, this problem may be formulated as

$$f(x) = x^2 - 10 = 0 \quad (7.15)$$

According to Newton's method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} = x_n - \frac{x_n^2 - 10}{2x_n} \quad (7.16)$$

which may be simplified to

$$x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n} \quad (7.17)$$

If the first approximation of the root is ten, then

$$\begin{aligned} X_2 &= \frac{10}{2} + \frac{5}{10} = 5.5000 \\ X_3 &= \frac{5.5000}{2} + \frac{5}{5.5000} = 3.6591 \\ X_4 &= \frac{3.6591}{2} + \frac{5}{3.6591} = 3.1960 \\ X_5 &= \frac{3.1960}{2} + \frac{5}{3.1960} = 3.1625 \\ X_6 &= \frac{3.1625}{2} + \frac{5}{3.1625} = 3.1623 \\ X_7 &= \frac{3.1623}{2} + \frac{5}{3.1623} = 3.1623 \end{aligned} \quad (7.18)$$

Therefore,

$$\sqrt{10} = 3.1623.$$

It is interesting to note that this procedure is used by the handheld calculators.

False Position: The method of false position is probably the most common numerical method used for solving difficult non-linear equations in one unknown by the digital computer. This is because the method of false position combines the reliability of the trial and error method with the efficiency of more sophisticated methods. Like the trial and error method, the method of false position requires stepping along the x-axis until a sign change in $f(x)$ is found. In Figure 7 this occurs between x_1 and x_2 . When the root is so bracketed, the point $[x_1, f(x_1)]$ becomes a pivot point denoted by P through which a number of lines will pass. Each additional line is used to make a better approximation of the root. The first line is drawn to $[x_2, f(x_2)]$, and the root is approximated to be where this line crosses the x-axis. In the figure this intersection is at x_3 . The approximation in the root is improved by drawing a second line through $[x_3, f(x_3)]$ and finding its intersection with the x-axis. This process is continued until the change in successive approximations of x falls below the prescribed error criterion.

The equation for calculating the new approximation x_{n+1} from point P and the current approximation is developed using Figure 8. Since the two triangles in the figure are similar,

$$\frac{f(x_p)}{x_p - x_{n+1}} = \frac{f(x_n)}{x_n - x_{n+1}} \tag{7.19}$$

Rearranging gives

$$x_{n+1} = \frac{x_p \cdot f(x_n) - x_n \cdot f(x_p)}{f(x_n) - f(x_p)} \tag{7.20}$$

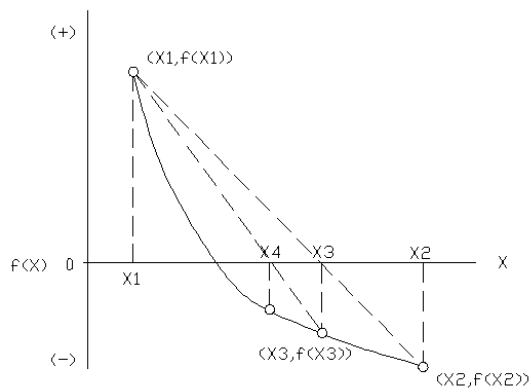


Figure 7-False Position Method

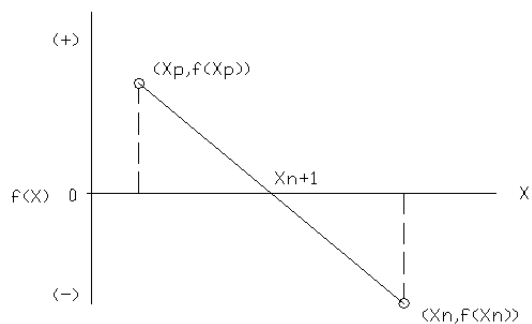


Figure 8-Derivation of the False Position Method Equation.

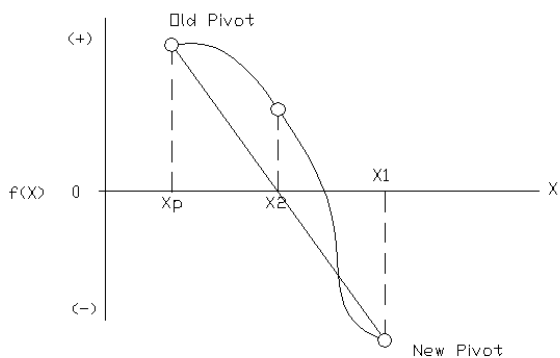


Figure 9-Changing the Pivot.

In some cases the shape of the curve $f(x)$ versus x will be such that $f(x_{n+1})$ will be the same sign as $f(x_p)$. This is shown in Figure 9. When this occurs, the pivot point must be relocated at $(x_n, f(x_n))$. Otherwise, the method degenerates into the secant method.

Example 7d

A steel ring is to be shrink fitted onto a steel shaft having a 3.010 inch diameter. The inside diameter of the steel ring is 3.000 inches. To what temperature must the ring be heated to fit over the shaft if the ring's diameter is given by

$$D = 3.000(1 + 0.81 \cdot 10^{-6}T + 0.72 \cdot 10^{-9}T^2 + 0.31 \cdot 10^{-12}T^3) \quad (7.21)$$

where T is the temperature in °F? Use the method of false position. The answer must be known to within one Fahrenheit degree.

Substitution of 3.010 for D in the above equation and rearranging gives

$$f(T) = \frac{3.010}{3.000} - 1 - 1.02 \cdot 10^{-6}T - 1.74 \cdot 10^{-9}T^2 - 0.39 \cdot 10^{-12}T^3 = 0 \quad (7.22)$$

Beginning at zero degrees Fahrenheit and stepping along the T -axis in 300 °F steps gives the following results:

T	$f(T)$
0	$3.33 \cdot 10^{-3}$
300	$2.86 \cdot 10^{-3}$
600	$2.01 \cdot 10^{-3}$
900	$7.22 \cdot 10^{-3}$
1200	$-1.07 \cdot 10^{-3}$

Therefore, the root is between 900 and 1200. If $(900, 7.22 \cdot 10^{-3})$ is made the pivot, and $(1200, -1.07 \cdot 10^{-3})$ is T_n , Eq. (7.20) gives

$$T_2 = \frac{(900)(-1.07 \cdot 10^{-3}) - (1200)(7.22 \cdot 10^{-4})}{(-1.07 \cdot 10^{-3}) - (7.22 \cdot 10^{-4})} \quad (7.23)$$

$$T_2 = 1020.87$$

Evaluating $f(T)$ at T_2 gives $6.37 \cdot 10^{-5}$ which has the same sign as the pivot. Therefore, the pivot must be changed to $(1200, -1.07 \cdot 10^{-3})$. Eq. (7.20) gives the next approximation as

$$T_3 = \frac{(1200)(6.37 \cdot 10^{-5}) - (1020.87)(-1.07 \cdot 10^{-3})}{(6.37 \cdot 10^{-5}) - (-1.07 \cdot 10^{-3})} \quad (7.24)$$

$$T_3 = 1030.96$$

Evaluating $f(T)$ at T_3 gives $4.98 \cdot 10^{-6}$ which has the opposite sign as the pivot. Therefore, the existing pivot may be used for the next approximation.

$$T_4 = \frac{(1200)(4.98 \cdot 10^{-6}) - (1030.96)(-1.07 \cdot 10^{-3})}{(4.98 \cdot 10^{-6}) - (-1.07 \cdot 10^{-3})} \quad (7.25)$$

$$T_4 = 1031.7$$

Since T_4 is within one Fahrenheit degree of T_3 , the answer is 1031.7 °F. If additional iterations are made to improve the accuracy of the answer, one will find $T = 1031.74$ °F.

Summary Many non-linear equations in one unknown have two or more roots. Although an equation may have both real and imaginary roots, the imaginary roots are usually of no engineering significance. The numerical methods presented in the chapter may be used to determine one root at a time. If only one root has any engineering significance, special care must be taken to assure that the root determined is the root of interest. For example, if the root signifies the time required to achieve thorough mixing in a tank, then only the positive root has any significance. Digital computer programs can be written so that insignificant roots are avoided. This is usually done by checking the root to see if it is significant. If it is not, new starting points for the numerical method are specified.

One-Point Iteration Method

The one-point iteration method in its simplest form for a single equation rearranges the function

$$f(x) = 0 \quad (7.26)$$

with unknown roots in such a way that

$$x = g(x). \quad (7.27)$$

The root, the value that solves Eq.(7.27), is sought by assuming a starting value of x to be used in $g(x)$ and computing a successive value for x such that

$$x_{N+1} = g(x_N). \quad (7.28)$$

This iterative procedure continues until convergence is reached, which is achieved if $|g'(x)| < 1$. A graphical view of this procedure is shown in Figure 10.

7.2 Simultaneous Linear Equations

Two methods of solving a system of linear equations are presented in this section: Gauss-Jordan Elimination and Gauss-Seidel.

Gauss-Jordan Elimination: One of the most efficient numerical methods of solving simultaneous linear equations is by the Gauss-Jordan Elimination Procedure. This procedure is most easily described by using it to solve a set of simultaneous linear equations such as

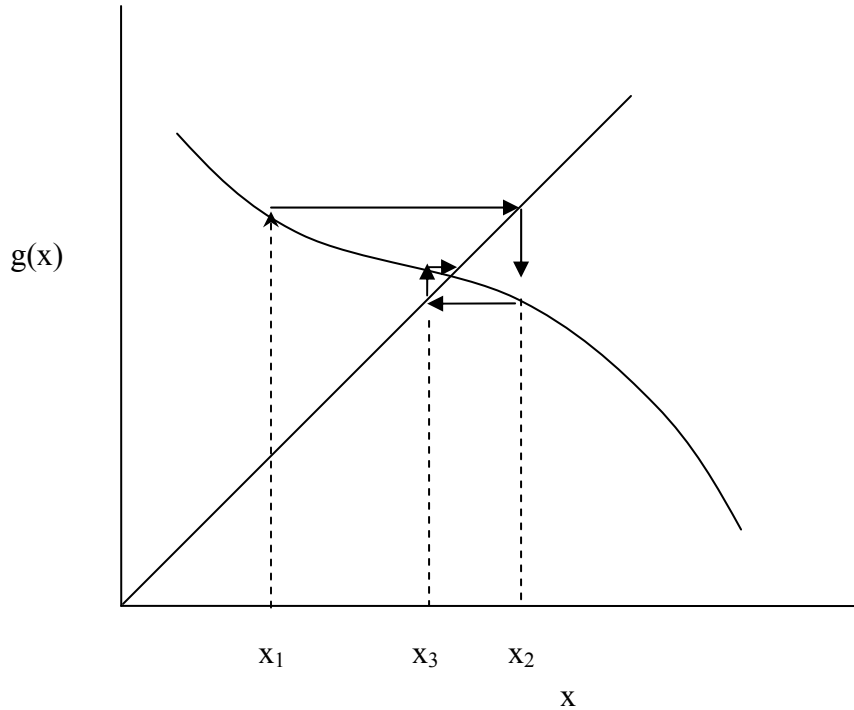


Figure 10-The One-Point Iteration Method

$$x + 3y + 2z = 11 \quad (7.29)$$

$$2x + y = 6 \quad (7.30)$$

$$-x + 2y - 2z = 9 \quad (7.31)$$

However, before using the Gauss-Jordan Elimination Procedure it is helpful to find the solution by inspection. Inspection shows that the addition of Eqs. (7.29) and (7.31) gives

$$y = 4 \quad (7.32)$$

Once y is known, Eq. (7.30) can be used to find that $x = 1$. Eq. (7.29) or (7.31) then gives $z = -1$. This solution by inspection provides a base from which the Gauss-Jordan elimination procedure may be explained. The Gauss-Jordan Method requires the use of a coefficient tableau like that shown in Tableau 7.1. Each value in the tableau represents a coefficient in Eq. (7.29) - (7.31). Since any solution of simultaneous linear equations involves only operations involving the coefficients, the tableau can be used to record the manipulations of the equations.

Tableau 7.1

x	y	z	
1	3	2	11
2	1	0	6
-1	2	-2	9

For example, the previous solution by inspection can proceed as follows. Inspection of this tableau shows that the addition of the third row to the first row eliminates the x and z coefficients in the first row to give

Tableau 7.2

x	y	z	
0	5	0	20
2	1	0	6
-1	2	-2	9

The first row could then be divided by five to give

Tableau 7.3

x	y	z	
0	1	0	4
2	1	0	6
-1	2	-2	9

This tableau corresponds to the equations

$$\begin{aligned} y &= 4 \\ 2x + y &= 6 \\ -x + 2y - 2z &= 9 \end{aligned}$$

Therefore, whenever all the variable coefficients in one row in a tableau are zero except for one which has the value of unity, the variable having the coefficient of unity equals the value at the end of the row. Consequently, the solution of simultaneous linear equations using the tableau consists of making all but one of the variable coefficients in each row equal to zero. The non-zero coefficient is made unity by dividing the entire row by its value. Applying this rationale to Tableau 7.3 requires that row one be subtracted from row two which is then divided by two to give

Tableau 7.4

x	y	z	
0	1	0	4
1	0	0	1
-1	2	-2	9

Therefore, $x = 1$. Continuing with the rationale requires that row two be added to row three and twice one be subtracted from row three to give

Tableau 7.5

x	y	z	
0	1	0	4
1	0	0	1
0	0	-2	2

Dividing row three by negative two gives

Tableau 7.6

x	y	z	
0	1	0	4
1	0	0	1
0	0	1	-1

Therefore, $z = -1$. The tableau in this exhibits a “basic solution”. That is, each variable column has one row with the coefficient of unity while all the other rows contain zeros.

The Gauss-Jordan Elimination Procedure involves the multiplication of rows in the coefficient tableau and the addition and subtraction of rows to generate a tableau with a basic solution. This procedure cannot rely on such fortuitous circumstances as the simultaneous elimination of both the x and z coefficients as was the case in the previous problem. The procedure consists of the steps below and will always result in a basic solution.

- 1) Choose any non-zero coefficient which is to become unity. All other values in this coefficient column are to become zero. This coefficient is called the pivot. The row and column containing the pivot are called the pivot row and pivot column, respectively.
- 2) Divide the pivot row by the pivot value and replace the old pivot row with the result.
- 3) Perform the following operation for each of the non-pivot rows. Multiply the pivot row by the coefficient in the pivot column of the non-pivot row and subtract the result from the non-pivot row. The pivot row remains unchanged, and the non-pivot row is left with a zero in the pivot column.
- 4) Select a new pivot not in a previous pivot row and repeat steps (2) and (3) until a basic solution is obtained.

Using the Gauss-Jordan Elimination Procedure to solve equations (7.7)-(7.9) beginning with Tableau 7.1 would proceed in the following manner. If the first coefficient in the x -coefficient column is selected as the pivot, step (2) is unnecessary. Step (3) yields the following tableau.

Tableau 7.7

x	y	z	
1	3	2	11
0	-5	-4	-16
0	5	0	20

If the second coefficient in the y -coefficient column is selected as the new pivot, the tableau becomes.

Tableau 7.8

x	y	z	
1	3	2	11
0	1	$4/5$	$16/5$
0	5	0	20

After step (2) and

Tableau 7.9

x	y	z	
1	0	-2/5	7/5
0	1	4/5	16/5
0	0	-4	4

after step (3). Making the last coefficient in the z-coefficient column the last pivot gives

Tableau 7.10

x	y	z	
1	0	0	1 = x
0	1	0	4 = y
0	0	1	-1 = z

Therefore, $x = 1$, $y = 4$, and $z = -1$ which is the same result obtained before.

Example 7d

A brass foundry has an order to cast 1000 brass valve bodies. The composition of the brass is to be 21.2 percent zinc, 77.14 percent copper, and 1.66 percent tin. The total weight of brass needed is 2000 pounds. The foundry foreman will make the 2000 pound melt by melting some scrap brass doorknobs containing 75 percent copper, 23 percent zinc, and two percent tin with some red brass containing 80 percent copper and 20 percent zinc. Also, copper with one percent tin will be used to make the 2000 pound melt. How much of each material should be taken to make the desired melt? All percentages are in weight percent?

Let x = pounds of brass scrap
 y = pounds of red brass
 z = pounds of copper-tin alloy.

The total pounds of copper, zinc, and tin in the final melt are 1542.8, 424.0, and 33.2, respectively.

Therefore,

$$\begin{aligned} 0.75x + 0.80y + 0.99z &= 1542.8 \\ 0.23x + 0.20y &= 424.0 \\ 0.02x &+ 0.01z = 33.2 \end{aligned}$$

The coefficient tableau is

Tableau 7.11

x	y	z	
0.75	0.80	0.99	1542.8
0.23	0.20	0	424
0.02	0	0.01	33.2

Making the first coefficient in the x-coefficient column the pivot gives

Tableau 7.12

x	y	z	
1	1.067	1.320	2057
0	-0.0454	-0.304	-49.11
0	-0.0213	-0.016	-7.940

Selection of the second coefficient in the y-column as the new pivot gives

Tableau 7.13

x	y	z	
1	0	-5.825	902.8
0	1	6.696	1081.7
0	0	0.1266	15.10

Finally, selection of the last coefficient in the z-column as the pivot gives

Tableau 7.14

x	y	z	
1	0	0	1597.7
0	1	0	282.9
0	0	1	119.3

Therefore, 1597.7 pounds of scrap brass, 282.9 pounds of red brass, and 119.3 pounds of copper-tin alloy are required.

Gauss-Seidel Method: The Gauss-Seidel Method, unlike Gauss-Jordan Elimination, is a numerical method that leads to an approximate solution. Gauss-Jordan Elimination is a direct method that leads to a solution that is as accurate as the computations employed. The Gauss-Seidel Method may be applied to a set of non-linear equations, but the outcome is not as certain as for linear systems.

The Gauss-Seidel Method proceeds as follows:

1. Arrange the equations to be solved so as to have a dominant diagonal. That is, the largest coefficients possible should fall on the diagonal of the coefficient matrix.
2. Each equation should be solved for the variable falling on the diagonal in terms of the other variables.
3. Initial values of the variables are arbitrarily selected for each variable.
4. A computational loop is continued in which each variable is successively computed from the equations solved in step 2 until the desired level of convergence is achieved.

The Gauss-Seidel Method will converge if the diagonal is dominant. Some systems cannot be arranged to have dominant diagonals because of the coefficients values. In such cases, some configurations of the diagonal may converge while others may not. Example 7e illustrates the use of the Gauss-Seidel Method.

Example 7e

Solve the following linear system using the Gauss-Seidel Method:

$$\begin{aligned} 2x - 6y - 4z &= -2 \\ 5x + 2y - 3z &= 14 \\ 3x + 3y + 8z &= 42 \end{aligned}$$

Rearrange the equations so that the largest coefficients lie on the diagonal as follows

$$\begin{aligned}5x + 2y - 3z &= 14 \\2x - 6y - 4z &= -2 \\3x + 3y + 8z &= 42\end{aligned}$$

This is said to have a dominant diagonal. Next, solve for each of the equations for the variables proceeding from left to right as follows:

$$\begin{aligned}x &= (14 - 2y + 3z)/5 \\y &= (-2 - 2x + 4z)/(-6) \\z &= (42 - 3x - 3y)/8\end{aligned}$$

Initial values (guesses) are selected for y and z (1 and 1, in this case), and updated estimates for x, y, and z are computed. The latest value for each variable is always used until the desired level of convergence is achieved as shown below.

When solving non-linear systems, the diagonal should contain the highest powers. Since there are multiple roots in non-linear systems, different initial values may lead to different roots.

Table 7. Progress of the Gauss-Seidel Solution

x	y	z
	1	1
3.0000	0.6667	3.8750
4.8583	-0.6306	3.6646
5.2510	-0.3594	3.4157
4.9932	-0.2794	3.4823
5.0012	-0.3212	3.4950
5.0255	-0.3215	3.4860
5.0202	-0.3173	3.4864
5.0187	-0.3180	3.4872
5.0195	-0.3183	3.4870

7.3 Systems of Non-Linear Equations

Presented here are two methods for solving a system of non-linear equations. The above Gauss-Seidel Method presented above may be used but the non-linearity may prevent convergence. Since there are multiple roots to a non-linear system of equations, the starting point can determine which roots are found if convergence is achieved. The likelihood of convergence is increased by arranging the equations in order of decreasing exponent power rather than decreasing coefficient as with linear equations. A matrix based solution to non-linear equations of great value is the Non-linear Newton-Raphson Method.

Multivariate Non-linear Newton-Raphson Method: This is one of the most efficient numerical methods for solving simultaneous non-linear equations. It is based on the same principle as the Newton's Method given in Eq. (7.13) for one equation in one unknown:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (7.12)$$

First iteration

$$x_2 := x - J(x)^{-1} \cdot f(x)$$

Current estimated roots

$$x_2 = \begin{pmatrix} 1.973 \\ 2.378 \\ 3.243 \end{pmatrix}$$

Second iteration

$$x_3 := x_2 - J(x_2)^{-1} \cdot f(x_2)$$

$$x_3 = \begin{pmatrix} 1.016 \\ 2.029 \\ 3.004 \end{pmatrix}$$

Third iteration

$$x_4 := x_3 - J(x_3)^{-1} \cdot f(x_3)$$

$$x_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Check answer

$$f(x_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 7g

Solve the linear system given in Example 7f using the *Multivariate Non-linear Newton-Raphson Method* and MATLAB.

Solution

Write the m file containing the function matrix and the Jacobian matrix and solve

```
x=[1 1 1 ]' % Set the initial guesses for x
for i= 1:5 % Start a loop for converging on the roots
    %Define the function matrix
    Jf=[x(1)+x(2)^2-3*x(3)+4;4*x(1)-x(2)+x(3)^2-11;x(2)^2+x(3)-7]
    %Define the Jacobian matrix
    JJ=[1, 2*x(2), -3 ; 4,-1,2*x(3);0,2*x(2), 1]
    %Compute the new estimates of the roots
    x=x-inv(JJ)*Jf
end %Repeat 5 times for convergence
```

1.0000	4.0585	-3.0000
4.0000	-1.0000	6.0079
0	4.0585	1.0000

x =

1.0000
2.0002
3.0000

Jf =

1.0e-03 *
0.8446
0.0154
0.8446

JJ =

1.0000	4.0004	-3.0000
4.0000	-1.0000	6.0000
0	4.0004	1.0000

x =

1.0000
2.0000
3.0000

Jf =

1.0e-07 *
0.4351
0.0010
0.4351

JJ =

1.0000	4.0000	-3.0000
4.0000	-1.0000	6.0000
0	4.0000	1.0000

x =

1.0000	
2.0000	
3.0000	Solution