4 – Boundary and Initial Conditions for Partial Differential Equations

In the previous chapter the boundary conditions have been the simplest of all possible boundary conditions: fixed temperature. In this chapter four other boundary conditions that are commonly encountered are presented for heat transfer: zero flux, fixed flux, convection, and radiation. Upon knowing how these conditions are applied in numerical solutions to arrive at heat conduction solutions, the student should be able to adapt the methods to essentially any boundary condition encountered in partial differential equations.

The following discussion, unless otherwise noted, assumes that a heat conduction in a solid problem is being solved. The partial differential equation to be solved involves a first partial of T with respect to t, time, and second partials of T with respect to position.

4.1 Fixed

This type of boundary condition is the simplest form of boundary condition in that one may enter a number for the value of the dependent value at the boundary as demonstrated in Chapter 3.

4.2 Zero Flux

A zero flux boundary condition requires that no heat flows across the zero-flux boundary. This condition is encountered when the surface is perfectly insulated or when it is the plane of symmetry. In the case of symmetry, the temperature is a maximum (on cooling) or a minimum (on heating) along the plane of symmetry. Regardless of the reason for the boundary being a zero-flux boundary, Fourier's Law of Heat Conduction requires the first derivative of T with respect to the direction perpendicular to the zero flux surface to be zero.

$$q_x = -k \left.\frac{\partial T}{\partial x}\right|_B = 0 \tag{4.1}$$

If the first derivative is zero then the temperature for the boundary may be set equal to the temperature one step inward from the surface. Of course, this is an approximation as are all numerical solutions. Therefore if the zero-flux surface is at x=0, then the boundary temperature must be

$$\left.T\right|_B = \left.T\right|_{\Delta x} \tag{4.2}$$

4.3 Fixed Flux

A fixed flux surface is commonly encountered when the solid is generating heat (electrical resistance heating, nuclear decay, induction heating, etc.) that eventually (at SS) must be removed from the solid. It does so by reaching a high enough temperature gradient at the surface such that the flux at the surface, as determined by Fourier's Law of Heat Conduction, conducts heat from the solid at the same rate as the heat is generated. In this discussion, the surface temperature will be fixed to assure the specified heat conduction at the surface as fixed by Fourier's Law of Heat Conduction.

$$q_{fixed} = -k \frac{\partial T}{\partial x} \approx -k \frac{T_{\Delta x} - T_B}{\Delta x} \tag{4.3}$$

Solving for the boundary temperature gives

$$T_B = T_{\Delta x} + \frac{q_{fixed}}{k}\Delta x \tag{4.4}$$

Clearly the zero flux case is a special case of the fixed flux boundary condition. That is, when the fixed flux is zero, Eqs. (4.3) and (4.4) are the same.

4.4 Convection

Perhaps the most common boundary condition encountered is the convection boundary condition. Heat lost from a solid (or liquid) surface may be described as follows:

$$q_{conv} = h(T_S - T_f) \tag{4.5}$$

where h is the heat transfer coefficient (given in this course) and T_f is the temperature of the fluid exchanging heat with the solid. Since the boundary surface cannot accumulate any energy as shown in Fig 4.1,

$$q_{conv} = q_{cond} \tag{4.6}$$

Substituting Eq. (4.5) and Fourier's Law of Heat Conduction into Eq. (4.6), and solving for T_B gives the desired equation for the boundary condition.

$$T_B = \frac{T_f h + T_{\Delta x}\left(k/\Delta x\right)}{h + \left(k/\Delta x\right)} \tag{4.7}$$

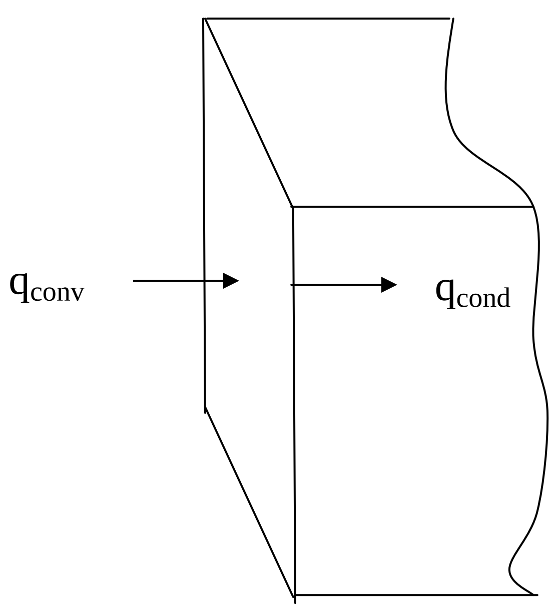

Figure 4.1. Heat balance for a surface during convective heat transfer

4.5 Radiation

Radiation flux from a hot surface is given by the Stephan-Boltzmann Equation

$$q_{Rad} = \varepsilon\sigma(T_B^4 - T_{Surr}^4) \tag{4.8}$$

where ε is emissivity, σ is the Stephan-Boltzmann Constant, and T_{Surr} is the temperature of the region the surface "sees". As before,

$$q_{Rad} = q_{cond} \qquad (4.9)\backslash$$

but unlike for convection this equation cannot be explicitly solved for T_B. Therefore, a radiation boundary condition requires a numerical root finding routine as discussed in the chapter on root finding.

4.6 Summary

Table 4.1 summarizes the equations to be placed at the boundary for each of the above five conditions.

Table 4.1. Summary of boundary condition for heat transfer and the corresponding boundary equation

Condition	**Equation**
Fixed	Any value (may vary)
Zero flux	$T_B = T_{\Delta x}$
Fixed flux	$T_B = T_{\Delta x} + \frac{q_{fixed}}{k}\Delta x$
Convection	$T_B = \frac{T_f h + T_{\Delta x}(k/\Delta x)}{h + (k/\Delta x)}$
Radiation	Solve for T_B numerically $-k\frac{T_{\Delta x} - T_B}{\Delta x} = \varepsilon\sigma(T_B^4 - T_{Surr}^4)$

4.7 Notation

In the following chapters the notation shown in Table 4.2 is often used for the temperatures appearing above.

Table 4.2. Common notation for boundary equations

Notation in Chapter 4	**Alternate notation**
T_B	T_S
$T_{\Delta x}$	T_{S-}

