3 - Elementary Solutions to Partial Differential Equations

Most students believe that the term *elementary solutions to partial differential equations* is either an oxymoron or an irritating erudite expression better restated when dealing with college students. It seems there should be nothing elementary about solving a partial differential equation. Therefore, it bears repeating that for a given partial differential equation – preferably one the student has derived and understands – it is a fairly simple algebraic matter to generate a solution to the equation using a worksheet. Of course, the solution gets more difficult as the number of independent variables increases. For example, the two directions on a worksheet (up-down and right-left) could be used to represent two independent variables. Figure 1 is a guide showing schematically how a worksheet may be construed for selected partial differential solutions.

Typically, in one-dimensional unsteady state heat conduction problems (1D USS HC), time is made to increase down the sheet while the position through the solid is represented across the sheet. This forms a table in which the computed temperatures at any time and position may be reported. In a two-dimensional steady state heat conduction problem (2D SS HC), the two directions on the sheet are used to represent the two dimensions in the conduction problem.

A two dimensional unsteady state heat conduction problem (2D USS HC) has three independent variables (t, x, y). There is no way to represent all of these on the two-dimensional field offered by the worksheet; therefore, some accommodation must be made. Typically this accommodation consists of using the two dimensions on the worksheet for the two independent dimensions (x and y). Each panel of temperatures on this area represents a particular time during the solution starting at the initial time and progressing to the final time as desired by the user. A three dimensional problem must be displayed as numerous slices through the solid be the solution steady state or unsteady state.

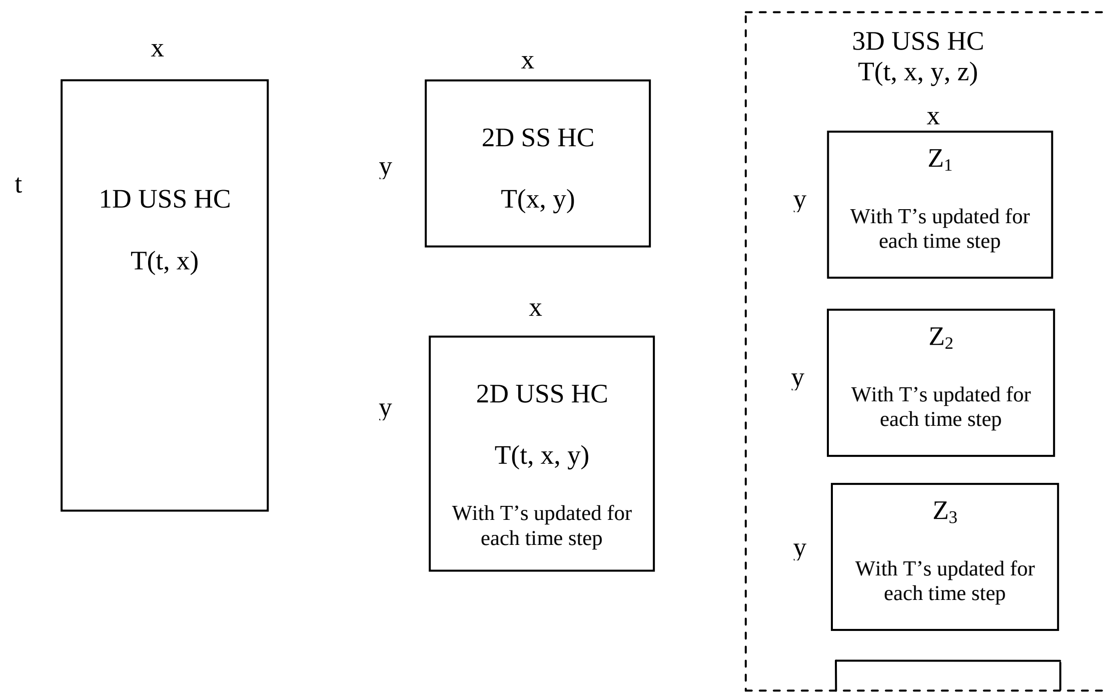


Figure 1. Guide to worksheet layout for selected partial differential equation solutions

3.1 - Partial Derivatives

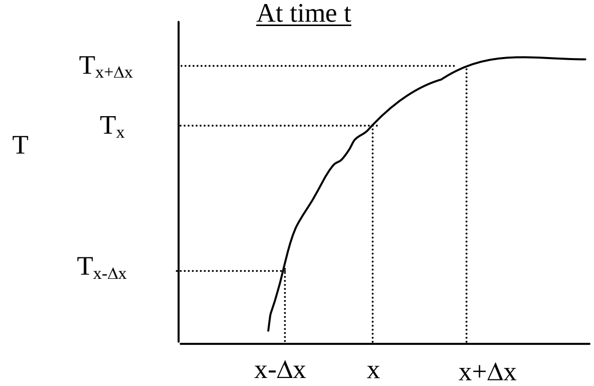
Whenever a dependent variable depends on more than one independent variable, the partial derivative is used to denote how the dependent variable changes with a particular variable. For example, if T(t, x) then the partial derivative of T with respect to t may be approximated as follows:

Forward: $$\frac{\partial T}{\partial x} \approx \frac{T_{t,x+\Delta x} - T_{t,x}}{\Delta x}$$

Backward: $$\frac{\partial T}{\partial x} \approx \frac{T_{t,x} - T_{t,x-\Delta x}}{\Delta x}$$

Central: $$\frac{\partial T}{\partial x} \approx \frac{T_{t,x+\Delta x} - T_{t,x-\Delta x}}{2\Delta x}$$

At time t



Likewise for the second derivative of T with respect to x:

Central: $$\frac{\partial^2 T}{\partial x^2} = \frac{\frac{T_{t,x+\Delta x} - T_{t,x}}{\Delta x} - \frac{T_{t,x} - T_{t,x-\Delta x}}{\Delta x}}{\Delta x} = \frac{T_{t,x+\Delta x} - 2T_{t,x} + T_{t,x-\Delta x}}{\Delta x^2}$$

Forward: $$\frac{\partial^2 T}{\partial x^2} = \frac{\frac{T_{t,x+2\Delta x} - T_{t,x+\Delta x}}{\Delta x} - \frac{T_{t,x+\Delta x} - T_{t,x}}{\Delta x}}{\Delta x} = \frac{T_{t,x+2\Delta x} - 2T_{t,x+\Delta x} + T_{t,x}}{\Delta x^2}$$

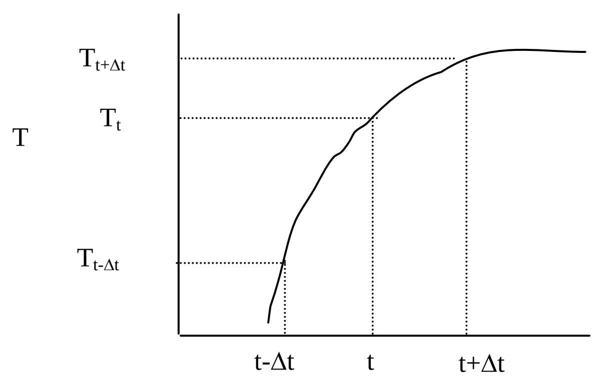
Backward: $$\frac{\partial^2 T}{\partial x^2} = \frac{\frac{T_{t,x} - T_{t,x-\Delta x}}{\Delta x} - \frac{T_{t,x-\Delta x} - T_{t,x-2\Delta x}}{\Delta x}}{\Delta x} = \frac{T_{t,x} - 2T_{t,x-\Delta x} + T_{t,x-2\Delta x}}{\Delta x^2}$$

The first order derivatives of T with respect to t are as follows:

Forward: $$\frac{\partial T}{\partial t} \approx \frac{T_{t+\Delta t,x} - T_{t,x}}{\Delta t}$$

Backward: $$\frac{\partial T}{\partial t} \approx \frac{T_{t,x} - T_{t-\Delta t,x}}{\Delta t}$$

Central: $$\frac{\partial T}{\partial t} \approx \frac{T_{t+\Delta t,x} - T_{t-\Delta t,x}}{2\Delta t}$$



3.2 - Shorthand Notation for First and Second Partial Derivatives in MATH 373

The previous notation style for derivative approximations is too cumbersome. A shorter notation is shown below. The position at x + Δx is denoted with a plus subscript while the position at x-Δx is denoted with a negative subscript. The position at x + 2Δx is denoted with a "+2" subscript. A "-2" is used to denote the position x - 2Δx. The time at t + Δt is denoted with a superscript prime mark *after* the dependent variable while the time at t - Δt is denoted with a prime superscript *before* the dependent variable. No notation is needed for the location at x or the time t. The previous full notation derivatives are now compared with the shorthand notation.

Time First Derivatives:

	Full Notation	Shorthand Notation	
Forward:	$\frac{\partial T}{\partial t} \approx \frac{T_{t+\Delta t} - T_t}{\Delta t}$	$\frac{\partial T}{\partial t} \approx \frac{T' - T}{\Delta t}$	(3.1)
Backward:	$\frac{\partial T}{\partial t} \approx \frac{T_t - T_{t-\Delta t}}{\Delta t}$	$\frac{\partial T}{\partial t} \approx \frac{T - {}'T}{\Delta t}$	(3.2)
Central:	$\frac{\partial T}{\partial t} \approx \frac{T_{t+\Delta t} - T_{t-\Delta t}}{2\Delta t}$	$\frac{\partial T}{\partial t} \approx \frac{T' - {}'T}{2\Delta t}$	(3.3)

Position Second Derivatives:

	Full Notation	Shorthand Notation	
Central:	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{t,x-\Delta x} - 2T_{t,x} + T_{t,x+\Delta x}}{\Delta x^2}$	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_- - 2T + T_+}{\Delta x^2}$	(3.4)
Forward:	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{t,x+2\Delta x} - 2T_{t,x+\Delta x} + T_{t,x}}{\Delta x^2}$	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{+2} - 2T_+ + T}{\Delta x^2}$	(3.5)
Backward:	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{t,x} - 2T_{t,x-\Delta x} + T_{t,x-2\Delta x}}{\Delta x^2}$	$\frac{\partial^2 T}{\partial x^2} \approx \frac{T - 2T_- + T_{-2}}{\Delta x^2}$	(3.6)

3.3 - Spreadsheet Solution of 1D USS HC Problems

The Heat Equation for a 1D USS HC problem with no generation term is

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{3.7}$$

Substitution of the finite approximations of the partials gives

$$\alpha \frac{T_{-x} - 2T + T_{+x}}{\Delta x^2} = \frac{T' - T}{\Delta t} \tag{3.8}$$

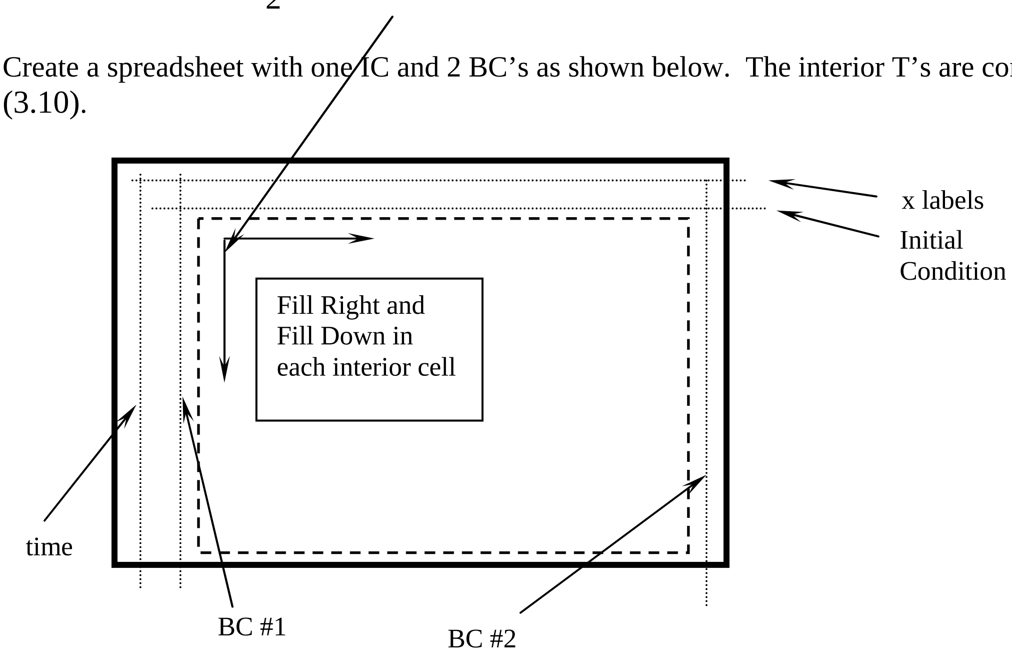
Solving for T' gives

$$T' = T + \frac{a\Delta t}{\Delta x^2}\left[T_{-x} - 2T + T_{+x}\right] \tag{3.9}$$

If the time step, Δt, is selected such that $\frac{a\Delta t}{\Delta x^2} = \frac{1}{2}$, then

$$T' = \frac{1}{2}\left[T_{-x} + T_{+x}\right] \tag{3.10}$$

Create a spreadsheet with one IC and 2 BC's as shown below. The interior T's are computed from Eq (3.10).



<u>3.4 - Spreadsheet Solution of 2D SS HC Problems</u>

The Heat Equation for a 2D SS HC problem with no generation term is

$$\alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] = 0 \tag{3.11}$$

Substitution of the finite approximations of the partials and forcing Δx = Δy gives

$$\frac{T_{-x} - 2T + T_{+x}}{\Delta x^2} + \frac{T_{-y} - 2T + T_{+y}}{\Delta x^2} = 0 \tag{3.12}$$

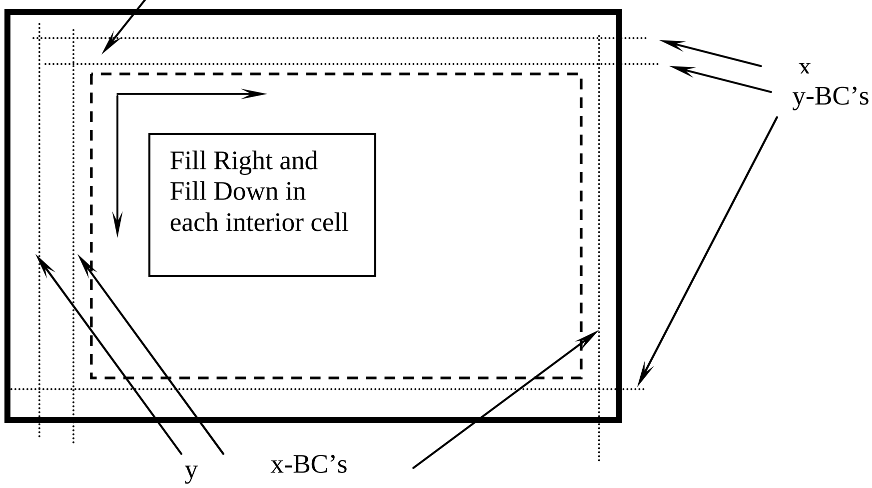
The student should note the use of plus and minus subscripts to denote the direction of Incrementation in the position derivatives.

Solving for T gives

$$T = \frac{1}{4}\left[T_{-x} + T_{+x} + T_{-y} + T_{+y}\right] \tag{3.13}$$

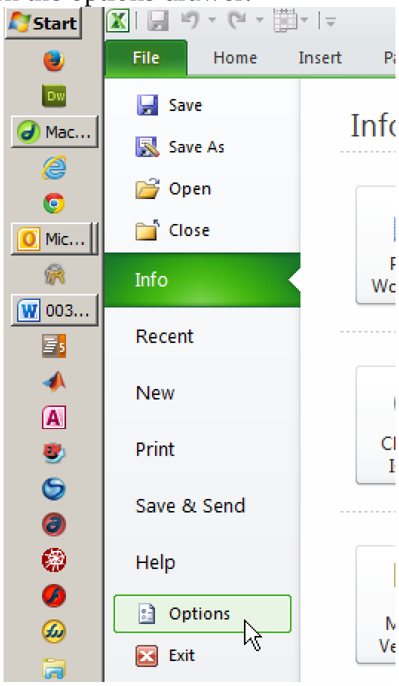
This equation may be used to "Relax" the solution into the steady state solution by repetitive computation. Be sure to set the spreadsheet to the "Iterative" mode to avoid circular references errors. See below.

Create a spreadsheet with two BC's in the x-direction and 2 BC's in the y-direction as shown below. The interior T's are computed from Eq. (3.13) by relaxation (repetition).

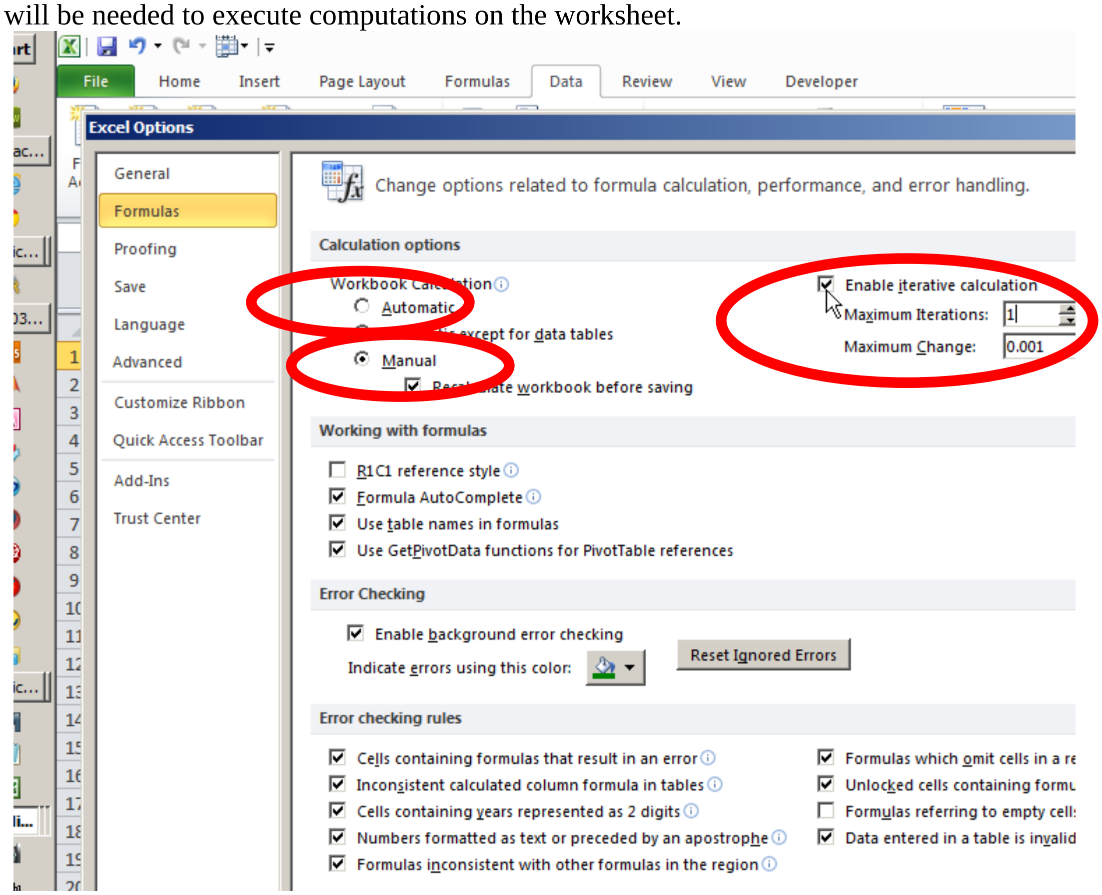


3.5 - Resolving the Circular Reference Error

1) Open the options drawer.



2) Select the Formulas menu item. Select Iteration, change from ‘100’ to ‘1’ iteration and select Manual. F9 will be needed to execute computations on the worksheet.



3.6 - Boundary Conditions

There are various boundary conditions encountered in actual heat transfer problems. The most common ones are described here.

Fixed Temperature

The fixed boundary condition is the simplest of all the boundary conditions. The temperature of the boundary is known. The equation at the surface for the temperature is simply

$$T = \text{fixed value} \tag{3.14}$$

Gradient

Gradient boundary conditions occur when the heat flux at the surface is known. In the simplest case this flux is zero. This condition represents either a perfectly insulated boundary or a boundary at which the geometry requires a minimum or maximum temperature. At the minimum or maximum, the slope of the temperature is zero. Therefore, there can be no heat flux. Another way to identify such a boundary condition is by symmetry. A mirror image temperature profile at a plane is a zero-flux plane.

Zero Flux: Zero flux is described as

$$q = 0 = -k\frac{dT}{dx} \tag{3.15}$$

Therefore,

$$\frac{dT}{dx} = 0 \tag{3.16}$$

at the zero-flux boundary. This condition can be satisfied by adding a “dummy” column next to the edge of the zero-flux boundary and making the temperatures in that column equal to the temperatures in the column on the opposite side of the zero boundary flux.

If the zero-flux boundary is at x=L then the temperature gradient may be approximated using the central difference approximation

$$\frac{dT}{dx} \approx \frac{T_{L+\Delta x} - T_{L-\Delta x}}{2\Delta x} = 0 \tag{3.17}$$

Therefore,

$$T_{L+\Delta x} = T_{L-\Delta x} \tag{3.18}$$

This is illustrated in the Figure 2. This is mathematically the same as setting the boundary to the temperature of the adjacent cell representing the inward temperature one step in from the zero-flux surface.

Fixed q: A fixed-q boundary condition is encountered when the rate of heat loss from a boundary is known. The temperature gradient at the fixed-flux boundary of the surface must be matched to the flux through Fourier’s Law of Heat Conduction. Therefore, for a 1D USS HC problem if the fixed-flux boundary is at x=L

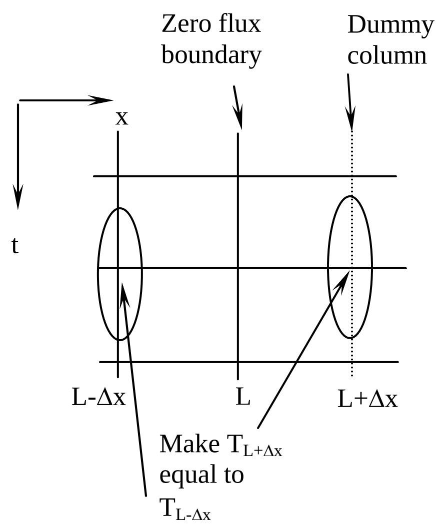


Figure 2. Zero-Flux Boundary Condition in a 1D USS HC Problem.

$$q_x = -k\frac{dT}{dx} \approx -k\frac{T_L - T_{L-\Delta x}}{\Delta x} \tag{3.19}$$

This gives the following equation at the boundary

$$T_L = T_{L-\Delta x} - \frac{q_x \Delta x}{k} \tag{3.20}$$

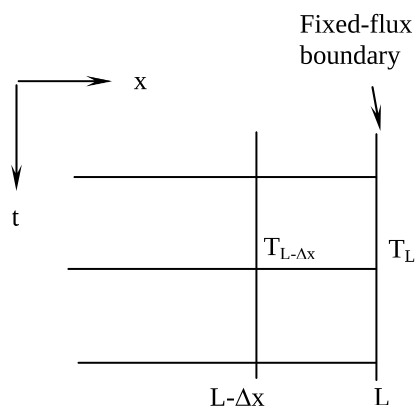


Figure 2. Fixed-Flux Boundary Condition in a 1D USS HC Problem.

Convection: A convection boundary condition is encountered whenever the solid loses heat by convection to a fluid. The rate of heat loss from a boundary is described by the equation

$$q_{conv} = h(T_L - T_f) \tag{3.21}$$

where h = the heat transfer coefficient (which will be known)
T_L = the temperature at the boundary surface
T_f = the temperature of the fluid

Since the surface at the boundary has no mass, it cannot store heat. Therefore the heat conducted to the surface must be equal to the heat lost from the surface by convection

$$q_{conv} = q_{cond} \tag{3.22}$$

substitution of Fourier's Law of Heat and Eq. (3.21) and solving for T_L gives an equation for the boundary.

$$T_L = \frac{T_f + cT_{L-\Delta x}}{1+c} = \tag{3.23}$$

where $c = \dfrac{k}{h\Delta x}$

Radiation: The loss by radiation from a surface is given by the equation

$$q_{rad} = \sigma\varepsilon(T_L^4 - T_S^4) \tag{3.24}$$

where σ = the Stephan –Boltzmann Constant = 5.5x10^{-12} Watts/(K^4*cm^2)
ε = emissivity (0 to 1; 0 for perfectly reflective surfaces, 1 for perfect absorbers)
T_S = the temperature of the surroundings

A radiation boundary condition, like the convection condition, requires that the flux to the surface by conduction must equal the flux lost by radiation heat transfer. Therefore,

$$q_{rad} = q_{cond} \tag{3.25}$$

substitution of Fourier's Law of Heat and Eq. (3.24) and solving for T_L gives a transcendental equation for the boundary temperature T_L

$$\sigma\varepsilon(T_L^4 - T_S^4) = -k\frac{T_L - T_{L-\Delta x}}{\Delta x} \tag{3.26}$$

Since this equation is transcendental, no explicit expression can be written for T_L. This complicates solving problems with radiation boundary conditions by requiring a numerical convergence (iteration) routine to solve for the boundary temperature. On a spreadsheet, this requires the use of a macro function that employs a root finding method.

3.7 - Spreadsheet Solution of a 2D USS HC Problem

The governing equation for a 2D USS HC problem with no generation is

$$\alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] = \frac{\partial T}{\partial t} \tag{3.27}$$

Substituting in the central difference approximations for the second order partial terms and the forward difference approximation for the first order partial and solving for the forward T gives

$$T^{'} = T + \frac{\alpha \Delta t}{\Delta x^2}\left[\left(T_{+x} - 2T + T_{-x}\right) + \left(T_{+y} - 2T + T_{-y}\right)\right] \tag{3.28}$$

The student should note the use plus and minus subscripts to denote the direction of incrementation in the position derivatives.

If Δt is set at a value such that $\frac{\alpha \Delta t}{\Delta x^2} = \frac{1}{4}$, then Eq. (3.28) becomes

$$T^{'} = \left[\frac{T_{+x} + T_{-x} + T_{+y} + T_{-y}}{4}\right] \tag{3.29}$$

It is very important to recognize that this looks like the same result as for a 2D SS HC problem but that it is different in that the left side is T', not simply T. The calculation of a new temperature using Eq.(3.29) requires that all new temperatures are computed from ALL old temperatures. In the 2D SS HC solution the "old" and "new" temperatures appearing during relaxation had nothing to do with time.

Therefore, to solve a 2D USS HC problem using Eq. (3.29) will require a panel of "old" T's from which all new T's are computed. Once all the new T's are computed for all positions in x and y, those T's become the old T's another panel of new T's is computed. If a new panel is used for each time step, then a very large number of panels would be needed to complete a solution. The need for many panels can be reduced to just two panels if, rather than using a third panel for the second computation, the first panel is used to hold the results for the second time step. The second panel could then be used to hold the results for the third time step and so on for as many time steps as desired. Figure 3 summarizes the calculation scheme.

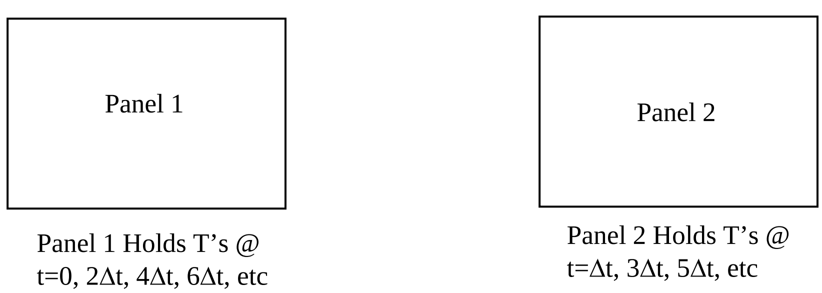


Figure 3. Calculation Scheme

Toggle and s

If one chooses to use a worksheet to solve a 2D USS HC problem, some means of controlling when the left or right panel computes is required. A variable *s* is used to reset the spreadsheet in case something goes wrong and the cells fill up with "REF" or other text indicating a problem. With a means of resetting the cell value, everything must be retyped and even then there is no guarantee the same problem will necessitate another retyping: a very discouraging prospect. Of course if one chooses to write code (VBA, MATLAB), no such 'resetting' procedure is needed if an error occurs.

Whenever *s* is set to 0, the spreadsheet cell values return to the initial temperatures as defined by the initial condition and the value of L is reset to 0. When *s* is set to a value other than 0, L will alternate between 0 and 1 each calculation cycle. This is accomplished with the following statement defining L (that is, the name of the variable L is created using the create variable menu command) to be

=IF(s, IF (L, 0, 1), 0) (3.30)

The value of L is used to turn on and off calculations on Panel 1 (left) and 2 (right). A typical statement for Panel 1, cell D10 would be

=IF(s, IF(L,(AD9+AD11+AC10+AE10)/4, D10), IC) (3.31)

assuming that Panel 2 is one alphabetic cycle to the right of Panel 1. A similar statement is used for cell AD10.

=IF(s, IF(L,AD10,(D9+D11+C10+E10)/4), IC) (3.32)

A time counter must be included. The cell holding the equation for the created name for time, t, would have the form

=IF(s, t+Δt, 0). (3.33)

Boundary Conditions
The edges of each panel are determined by the four Boundary Conditions.

3.8 – When Thermal Conductivity is a Linear Function of Temperature

The derivation of the partial differential equation for one-dimensional unsteady state heat transfer begins with the one-dimensional heat balance.

$$-\frac{\partial q_x}{\partial x} = \rho Cp \frac{\partial T}{\partial t} \tag{3.34}$$

Substituting Fourier's Law for q_x gives

$$-\frac{\partial\left(-k\left(\frac{dT}{dx}\right)\right)}{\partial x} = \rho Cp \frac{\partial T}{\partial t} \tag{3.35}$$

Making k a linear function of temperature gives

$$\frac{\partial\left((a+bT)\left(\frac{dT}{dx}\right)\right)}{\partial x} = \rho Cp \frac{\partial T}{\partial t} \tag{3.36}$$

which becomes

$$(a+bT)\left(\frac{d^2T}{dx^2}\right) + b\left(\frac{dT}{dx}\right)^2 = \rho Cp \frac{\partial T}{\partial t} \tag{3.37}$$

Estimating the derivatives using the central differences for the partials with respect to position and the forward difference for the time derivative gives

$$\left(a+bT\right)\left(\frac{T_{+}-2T+T_{-}}{\Delta x^{2}}\right)+b\left(\frac{T_{+}-T_{-}}{2\Delta x}\right)^{2}\approx \rho Cp\frac{T'-T}{\Delta t} \tag{3.38}$$

Solving for T′ provides the basis for the approximate numerical solution.

$$T'\approx T+\left(\frac{a+bT}{\rho Cp}\right)\Delta t\left(\frac{T_{+}-2T+T_{-}}{\Delta x^{2}}\right)+\frac{b\Delta t}{\rho Cp}\left(\frac{T_{+}-T_{-}}{2\Delta x}\right)^{2} \tag{3.39}$$

