2 – Owning Differential Equations

Differential equations are the currency of engineering and science. Without them our attempt to describe the world we see and the wonders we create – or strive to create – would be severely limited, because systems exhibiting rates of change, slopes, and curvature would be beyond our grasp to describe mathematically.

The great majority of time that technical students spend on calculus in their first two years of engineering or science at SDSM&T and other universities greatly matures their ways of thinking and their fundamental skills. However, few are confident about deriving differential equations that describe real systems. This is not a fault of the mathematics course design and content. In fact, the development of such skills is deliberately assigned to engineering and applied courses such as this one.

One might argue that some better coordination between engineer science and other applied courses with the calculus and ordinary differential sequence is needed. Such a suggestion was acted on by the faculty at SDSM&T during the 2003-4 academic year and slated for piloting during the 2004-5 academic year. The pilot model calls for shared classroom interaction between calculus and engineering science courses.

In this chapter, the student will learn to derive their own differential equations. The preferred setting is heat conduction where the equation describes temperature variations through a solid of constant thermal conductivity. These equations are fairly straight forward to derive and are useful to virtually all disciplines. Rectilinear, cylindrical, and spherical coordinate systems will be included; however, angular variations in cylindrical and spherical coordinates are typically not considered and left for more advanced courses on heat transfer.

In this chapter the student will learn

- the need for an energy balance and how to achieve one,
- the meaning and use of a flux,
- the generation of derivatives as the incremental dimensions approach zero, and
- the number of sufficient boundary and initial conditions to solve a differential equation.

2.1 - Approximation of the First and Second Partial Derivatives from Slope Considerations

Taylor Series approximation may be rearranged to obtain approximations. The same approximations can be obtained from basic slope considerations; however, such an approach does not offer any insight into the order of error.

First Derivative:

Forward: $$\frac{dT}{dx} \approx \frac{T_{x+\Delta x} - T_x}{\Delta x} \qquad (2.1)$$

Backward: $$\frac{dT}{dx} \approx \frac{T_x - T_{x-\Delta x}}{\Delta x} \qquad (2.2)$$

Central: $$\frac{dT}{dx} \approx \frac{T_{x+\Delta x} - T_{x-\Delta x}}{2\Delta x} \qquad (2.3)$$

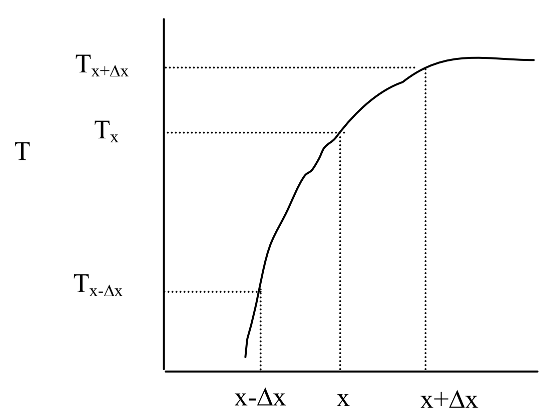


Figure 1 Variation of T with x

Second Derivative:

Likewise for the second derivative:

Central: $$\frac{d^2T}{dx^2} = \frac{\frac{T_{x+\Delta x} - T_x}{\Delta x} - \frac{T_x - T_{x-\Delta x}}{\Delta x}}{\Delta x} = \frac{T_{x+\Delta x} - 2T_x + T_{x-\Delta x}}{\Delta x^2} \quad (2.4)$$

Forward: $$\frac{d^2T}{dx^2} = \frac{\frac{T_{x+2\Delta x} - T_{x+\Delta x}}{\Delta x} - \frac{T_{x+\Delta x} - T_x}{\Delta x}}{\Delta x} = \frac{T_{x+2\Delta x} - 2T_{x+\Delta x} + T_x}{\Delta x^2} \quad (2.5)$$

Backward: $$\frac{d^2T}{dx^2} = \frac{\frac{T_x - T_{x-\Delta x}}{\Delta x} - \frac{T_{x-\Delta x} - T_{x-2\Delta x}}{\Delta x}}{\Delta x} = \frac{T_x - 2T_{x-\Delta x} + T_{x-2\Delta x}}{\Delta x^2} \quad (2.6)$$

2.2 - What is a Flux?

A flux is a flow per unit area and unit time. Table 1 shows how various flows may be reported as fluxes and their corresponding units.

Table 1. Fluxes

Flux Type	Law	Common Symbols and Equations	Corresponding units
mole	Fick's Law of Diffusion	$N_{A,x} = -\mathcal{D}_A \frac{\partial C_A}{\partial x}$	gmoles/cm^2 s
heat	Fourier's Law of Heat Conduction	$q_x = -k \frac{\partial T}{\partial x}$	Joules/m^2 s
momentum	Newton's Law of Viscosity	$\tau_{xy} = -\eta \frac{\partial V_y}{\partial x}$	force·velocity/m^2 s

Whenever a flux is multiplied by an area, a rate is obtained. Whenever a flux is multiplied by both area and time, an amount is obtained.

2.3 - Second partial notation and units

Tracking units is a useful way of checking terms obtained in derivations. To that end it is important to note the reason for the notation of the second derivative of x with respect to t. Can you explain which of the following forms for the second derivative is correct and why?

$$\frac{\partial^2 x}{\partial t^2} \qquad \frac{\partial x^2}{\partial t^2} \qquad \frac{\partial x^2}{\partial^2 t} \qquad \frac{\partial^2 x}{\partial^2 t} \quad (2.7)$$

It is a matter of units. If you think about the units of acceleration and acceleration's relationship to the second derivative of distance with respect to time, you will recall that

$$a[=]\frac{m}{\sec^2}. \quad (2.8)$$

That is, the dependent variable of distance is *not* squared but the independent variable of time *is* squared. It is so with every second derivative and higher-order derivative. Therefore, the order notation on a derivative is a matter of units.

2.4 - Fourier's Law of Heat Conduction

Fourier's Law of Heat Conduction arises from the experimental observation that heat flows from high to low temperatures by conduction. The rate of heat flow per area (heat flux q) depends on two things: 1) the temperature gradient in the direction of the flow and 2) the thermal conductivity, k, of the material.

$$q_x = -k\frac{\partial T}{\partial x} \tag{2.9}$$

The flow, or flux, is a vector with the concomitant direction and sense as shown in Figure 2. Since the thermal conductivity is always a positive scalar number and since heat flows down the temperature gradient, the sense of the heat flux is always opposite the sense of the temperature gradient. Thermal conductivity is the constant that relates the flux to the temperature gradient.

In incremental form, the heat flux is directly proportional to the temperature difference and indirectly proportional to the thickness.

$$-q_x \propto \frac{1}{L}\left(T_2 - T_1\right) \tag{2.10}$$

The thickness, L, is $(x_2\text{-}x_1)$

$$-q_x \propto \frac{\left(T_2 - T_1\right)}{\left(x_2 - x_1\right)} \tag{2.11}$$

Fourier's Law applies for any point along the x axis so it is written in the limit as $(x_2 - x_1)$ approaches zero. Therefore,

$$q_x = -k\frac{\partial T}{\partial x} \tag{2.12}$$

Since the distance term approaches zero in the derivative, Fourier's Law of Heat Conduction applies to cylindrical and spherical coordinates as well and would be written in terms of radial coordinates.

$$q_r = -k\frac{\partial T}{\partial r} \tag{2.13}$$

2.5 - Five Steps Needed to Derive a Differential Equation

Differential equations for the great majority of engineering systems may be derived by performing a balance on the quantity (heat, mass, charge, etc.) of interest. To correctly perform this balance one should perform the following five steps:

1. Draw a sketch.

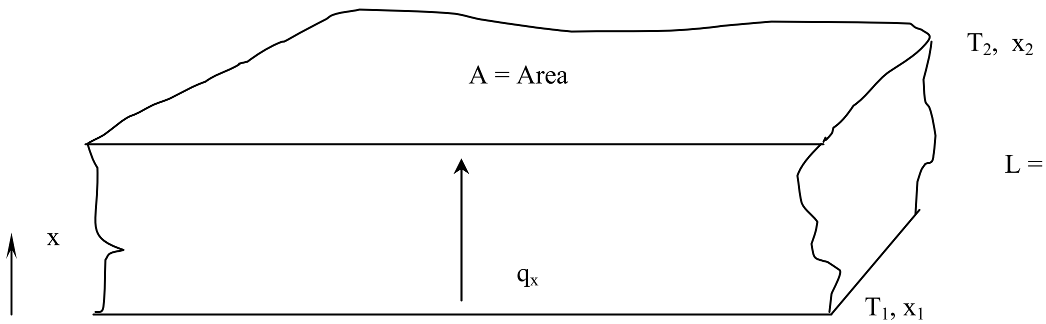


Figure 2. Heat Conduction in a Solid

2. Establish an incremental element that has an incremental thickness in each direction of change.
3. Perform a balance on the increment for the extensive quantity of interest (force, heat, etc.).
4. Divide through by all independent incremental terms (Δx, Δy, Δt, etc. – not the dependent variable such as ΔT) and take the limit as each goes to zero to obtain differential terms.
5. Substitute for any flux terms (q, τ, etc.) the fundamental law (Fourier's Law of Heat Conduction, Newton's Law of Viscosity, etc.) that relates these fluxes to intensive property gradients.

2.6 - Differential Equation for 2D USS HC without Generation

Derive a differential equation describing unsteady-heat conduction in 2 dimensions (2D USS HT)

Steps 1 and 2: Draw the sketch as shown in Figure 3. The direction of change is the x direction.

Step 3. Perform a Heat Balance on the element

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc}$$

$$\left[L\Delta y(q_{\mathrm{x}} - q_{x+\Delta x}) + L\Delta x(q_{\mathrm{y}} - q_{y+\Delta y}) + 0\right]\Delta t = \mathrm{L}\Delta X\Delta y\rho Cp(T_{t+\Delta T} - T_t) \tag{2.14}$$

Step 4. Divide by ΔtΔxΔyL

$$\frac{(q_{\mathrm{x}} - q_{x+\Delta x})}{\Delta x} + \frac{(q_{\mathrm{y}} - q_{y+\Delta y})}{\Delta y} = \rho Cp\frac{(T_{t+\Delta T} - T_t)}{\Delta t} \tag{2.15}$$

In the limit as each delta term approaches 0

$$-\frac{\partial q_{\mathrm{x}}}{\partial x} - \frac{\partial q_{\mathrm{y}}}{\partial y} = \rho Cp\frac{\partial T}{\partial t} \tag{2.16}$$

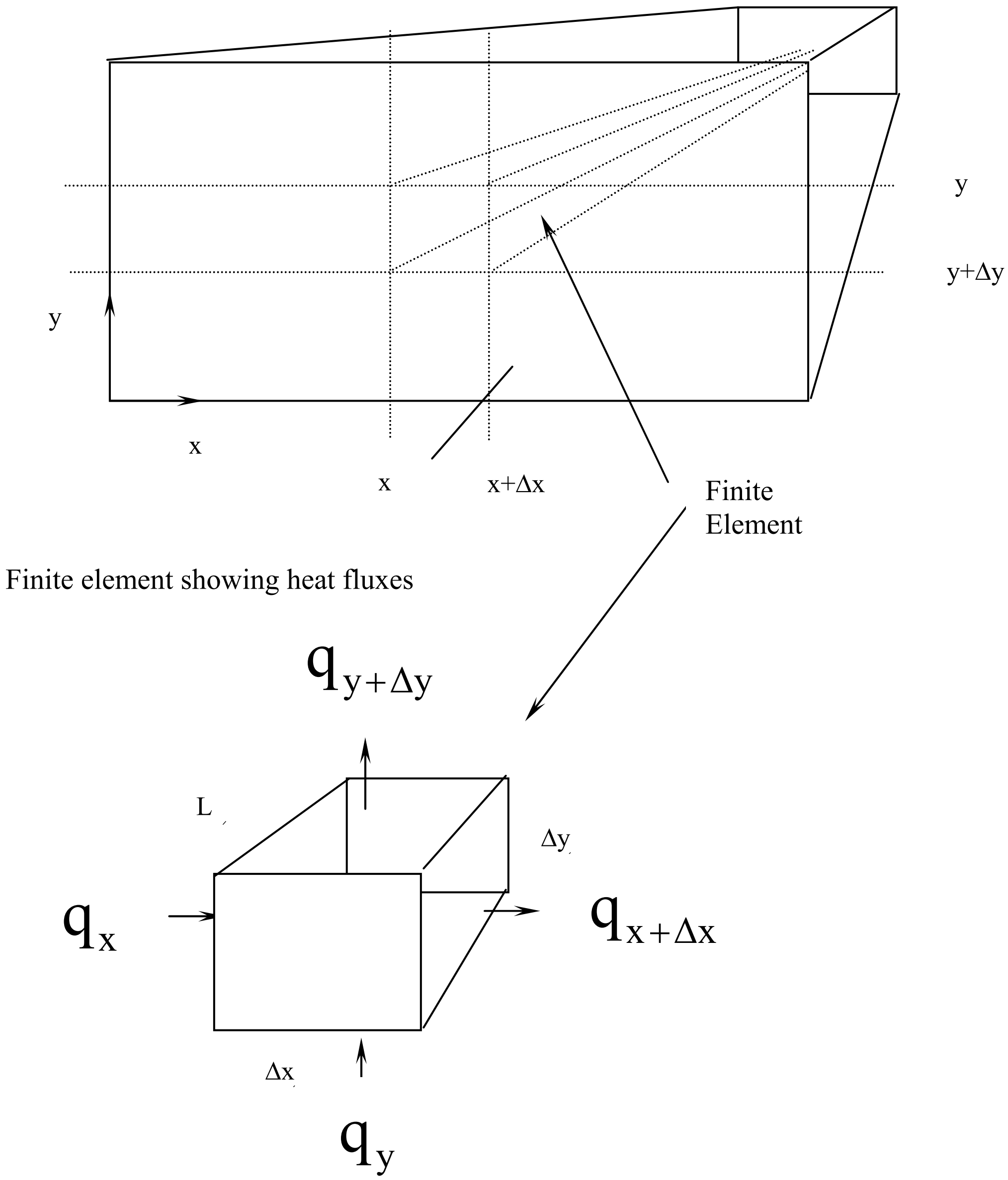


Figure 3. Sketch for the 2D USS HC Problem in Rectilinear Coordinates

Step 5. Substitute Fourier's Law of heat Conduction: $q_x = -k\dfrac{\partial T}{\partial x}$ and simplify

$$\alpha\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]=\frac{\partial T}{\partial t} \tag{2.17}$$

where $\alpha = \dfrac{k}{\rho Cp}$ (2.18)

2.7 - Differential Equation for 2D USS HC with Generation

In this derivation it will be assumed that a rate of energy generation per unit volume is S. It is uniform throughout the solid. Since the geometry is the same as the previous derivation, Figure 3 may also be used for this problem.

Steps: 1 and 2: Same as for the previous derivation.

Step 3. Heat Balance

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc}$$

$$\left[L\Delta y(q_{\text{x}} - q_{x+\Delta x}) + L\Delta x(q_{\text{y}} - q_{y+\Delta y}) + \text{SL}\Delta X\Delta y\right]\Delta t = \text{L}\Delta X\Delta y\rho Cp(T_{t+\Delta t} - T_t) \quad (2.19)$$

Step 4: Divide by ΔtΔxΔy and L

$$\frac{(q_{\text{x}} - q_{x+\Delta x})}{\Delta x} + \frac{(q_{\text{y}} - q_{y+\Delta y})}{\Delta y} + S = \rho Cp\frac{(T_{t+\Delta t} - T_t)}{\Delta t} \quad (2.20)$$

In the limit as each delta term approaches 0

$$-\frac{\partial q_{\text{x}}}{\partial x} - \frac{\partial q_{\text{y}}}{\partial y} + S = \rho Cp\frac{\partial T}{\partial t} \quad (2.21)$$

Step 5: Substitute Fourier's Law of Heat Conduction: $q_x = -k\frac{\partial T}{\partial x}$

$$\alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + \frac{S}{\rho Cp} = \frac{\partial T}{\partial t} \quad (2.22)$$

where $\alpha = \frac{k}{\rho Cp}$ (2.23)

2.8 - Differential Equation for 1D USS HC in Cylindrical Coordinates with No Generation

Steps: 1 and 2: Figure 5 shows the sketch for the derivation. The direction of change is r. Sketch and label

Step 3. Heat Balance

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc}$$

$$\left[2\pi L\left((rq)_{\text{r}} - (rq)_{r+\Delta r}\right) + 0\right]\Delta t = 2\pi r\text{L}\Delta r\rho Cp(T_{t+\Delta t} - T_t) \quad (2.24)$$

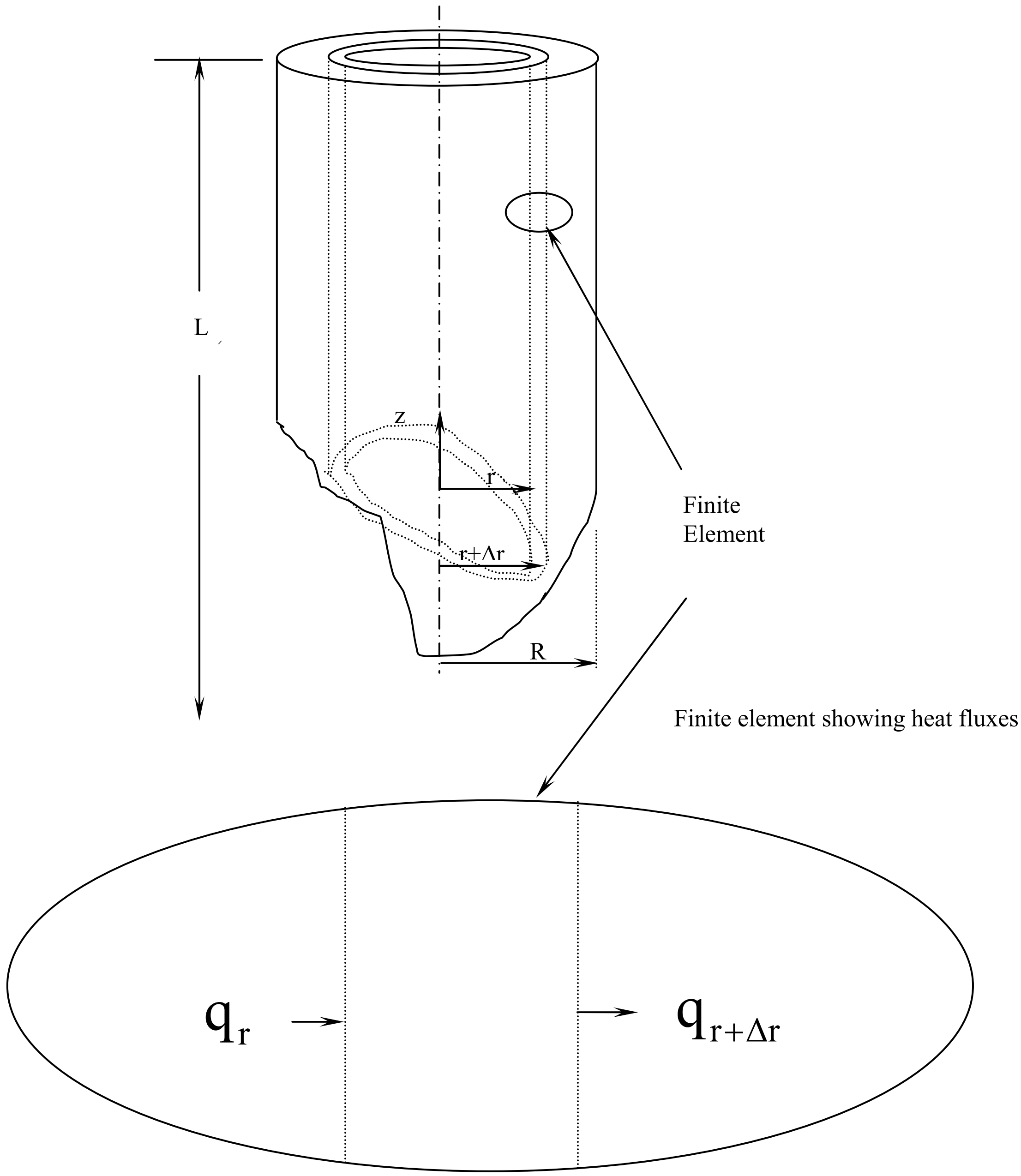


Figure 5. Sketch for a 1D USS HC Problem in Cylindrical Coordinates

Step 4. Divide by 2πLΔrΔt

$$\frac{(rq)_{\text{r}} - (rq)_{r+\Delta r}}{\Delta r} = r\rho Cp\frac{(T_{t+\Delta t} - T_t)}{\Delta t} \tag{2.25}$$

In the limit as each delta term approaches 0

$$-\frac{\partial rq_{\text{r}}}{\partial r} = r\rho Cp\frac{\partial T}{\partial t} \tag{2.26}$$

Step 5. Substitute Fourier's Law of Heat Conduction: $q_r = -k\frac{\partial T}{\partial r}$

$$\alpha\frac{\partial\ (rq_{\text{r}})}{\partial r} = r\frac{\partial T}{\partial t} \tag{2.27}$$

Expanding the first term and dividing by r gives

$$\alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] = \frac{\partial T}{\partial t} \tag{2.28}$$

where $\alpha = \frac{k}{\rho Cp}$ (2.29)

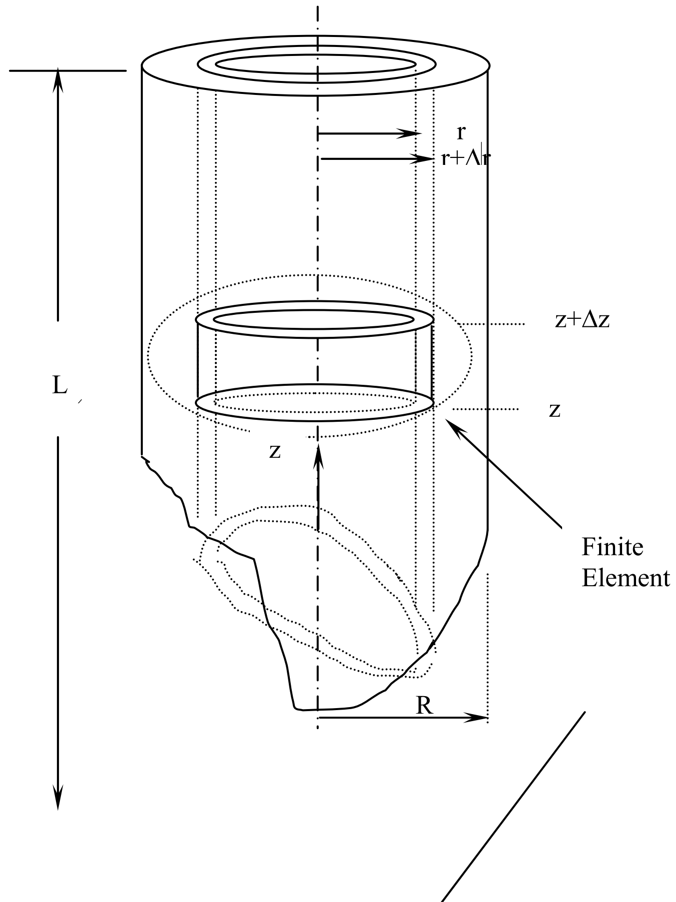
2.9 - Differential Equation for 2D USS HC in Cylindrical Coordinates with Generation

Steps 1 and 2: The 2 directions of change are r and z. Figure 6 shows the sketch of the derivation.

Step 3: Heat Balance

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc} \tag{2.30}$$

$$\begin{aligned}&\left[2\pi\Delta z\left((rq_r)_{\text{r}} - (rq_r)_{r+\Delta r}\right) + 2\pi r\Delta r\left((q_z)_{\text{z}} - (q_z)_{z+\Delta z}\right) + 2\pi r\Delta r\Delta zS\right]\Delta t \\ &= 2\pi r\Delta z\Delta r\rho Cp(T_{t+\Delta t} - T_t)\end{aligned} \tag{2.31}$$



Finite element showing heat fluxes

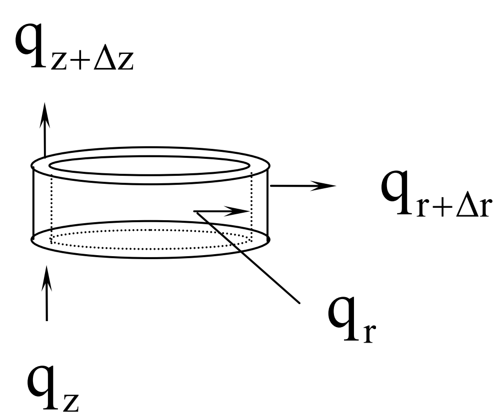


Figure 6. Sketch for the 2D USS HC Derivation in Cylindrical Coordinates

Step 4: Divide by $2\pi r \Delta r \Delta z \Delta t$

$$\frac{1}{r}\frac{(rq_r)_{\mathrm{r}} - (rq_r)_{r+\Delta r}}{\Delta r} + \frac{(q_z)_{\mathrm{z}} - (q_z)_{z+\Delta z}}{\Delta z} + S = \rho Cp\frac{(T_{t+\Delta t} - T_t)}{\Delta t} \tag{2.32}$$

In the limit as each delta term approaches 0

$$-\frac{1}{r}\frac{\partial rq_{\mathrm{r}}}{\partial r} - \frac{\partial q_{\mathrm{z}}}{\partial z} + S = \rho Cp\frac{\partial T}{\partial t} \tag{2.33}$$

Step 5: Substitute Fourier's Law of Heat Conduction: $q_r = -k\frac{\partial T}{\partial r}$ *and* $q_z = -k\frac{\partial T}{\partial z}$

$$\alpha\left[\frac{1}{r}\frac{\partial\ \ (r(\partial T/\partial r))}{\partial r} + \frac{\partial\ \ (\partial T/\partial z)}{\partial r} + \frac{S}{\rho Cp}\right] = \frac{\partial T}{\partial t} \tag{2.34}$$

Expanding the first term and dividing by r gives

$$\alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{S}{\rho Cp} = \frac{\partial T}{\partial t} \tag{2.35}$$

$$\text{where } \alpha = \frac{k}{\rho Cp} \tag{2.36}$$

2.10 - Differential Equation for 1D USS HC in Spherical Coordinates with Generation

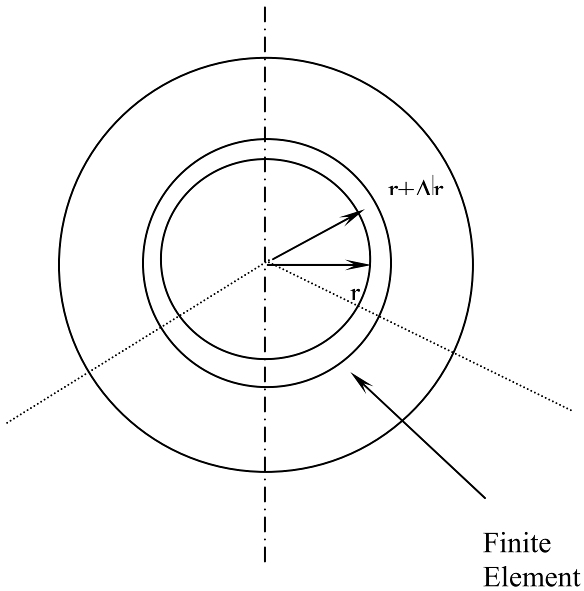
In this derivation it will be assumed that a rate of energy generation per unit volume is S. It is uniform throughout the solid.

Step1 and 2. The sketch for the derivation is shown in Figure 7.

Step 3. Heat Balance.

$$\mathrm{In} - \mathrm{Out} + \mathrm{Gen} = \mathrm{Acc}$$

$$\left[4\pi\left((r^2 q_r)_{\mathrm{r}} - (r^2 q_r)_{r+\Delta r}\right) + 4\pi r^2 \Delta r S\right]\Delta t = 4\pi r^2 \Delta r \rho Cp(T_{t+\Delta t} - T_t) \tag{2.37}$$



Finite element showing heat fluxes

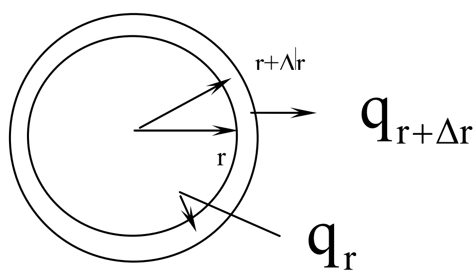


Figure 7. Sketch for the 1D USS HC Derivation

Step 4. Divide by $4\pi r^2 \Delta r \Delta t$ and take limit to get differential equation

$$\frac{1}{r^2}\frac{(r^2 q_r)_{\mathrm{r}} - (r^2 q_r)_{r+\Delta r}}{\Delta r} + S = \rho Cp \frac{(T_{t+\Delta t} - T_t)}{\Delta t} \tag{2.38}$$

In the limit as each delta term approaches 0

$$-\frac{1}{r^2}\frac{\partial\left(r^2 q_{\mathrm{r}}\right)}{\partial r} + S = \rho Cp \frac{\partial T}{\partial t} \tag{2.39}$$

Step 5. Substitute Fourier's Law of Heat Conduction: $q_r = -k\dfrac{\partial T}{\partial r}$

$$\alpha\left[\frac{1}{r^2}\frac{\partial\ (r^2(\partial T/\partial r))}{\partial r} + \frac{S}{\rho Cp}\right] = \frac{\partial T}{\partial t} \tag{2.40}$$

Expanding the first term and dividing by r gives

$$\alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right] + \frac{S}{\rho Cp} = \frac{\partial T}{\partial t} \tag{2.41}$$

$$\text{where } \alpha = \frac{k}{\rho Cp} \tag{2.42}$$

2.11 - Differential Equation for 1D SS Rate of Momentum in Cyl Coordinates

As a fluid flows smoothly through a cylinder, it may be thought of as an infinite series of concentric tubes of fluid. Each tube rubs against the adjacent tubes. This rubbing occurs because the inner fluid is traveling at a higher velocity than the fluid closer to the outer wall. Since the inner tubes are going faster than the outer tubes, each tubes inner surface receives a momentum boost from the rubbing. This is the basis of Newton's Law of viscosity.

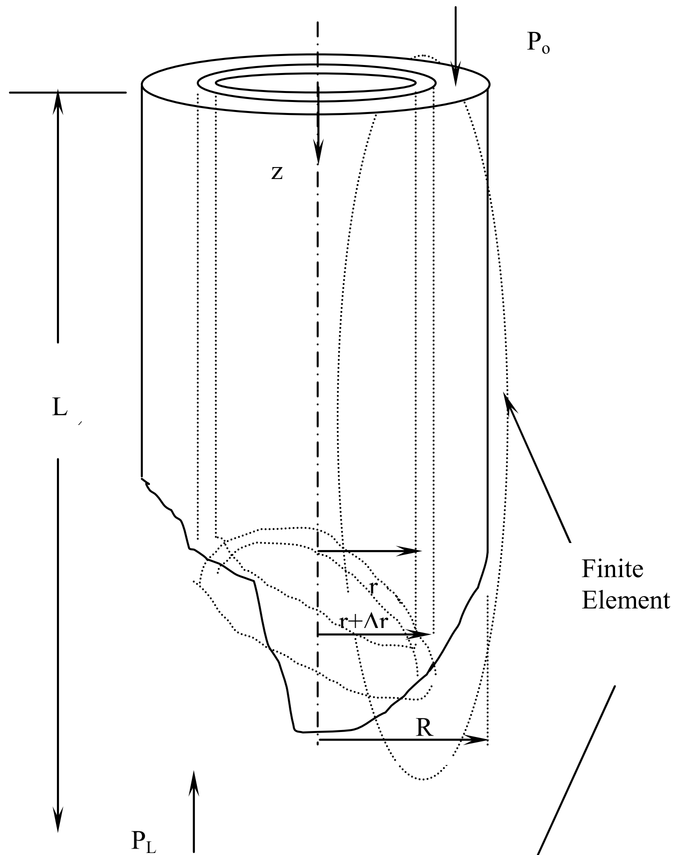
$$\tau_{rz} = -\eta\frac{\partial V_z}{\partial r} \tag{2.43}$$

where τ_{rz} is the rate of this momentum boost per unit area (a flux) in the z direction because of a variation in the radial direction (derivative) of velocity down the tube. The viscosity, η, is actually defined as the parameter that makes the rate of momentum flux equal to the velocity gradient. This is the same idea that determines the thermal conductivity in Fourier's Law of Heat Conduction.

$$q_r = -k\frac{\partial T}{\partial r} \tag{2.44}$$

As with all differential equation derivations, this one follows the same five steps except a heat balance is replaced by a rate of momentum balance. This is just a force balance on the element since rate of momentum is a force. Unless the body is accelerating, the sum of forces will equal zero. The steps remain the same: In this derivation it will be assumed that the direction z is down which is the same as the direction of gravity

Step 1 and 2. Figure 8 shows the sketch for this derivation. The direction of change is r.



Finite element showing momentum fluxes

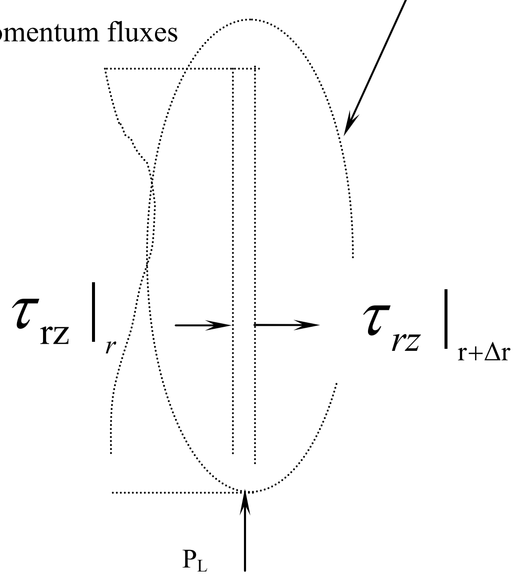


Figure 8. Sketch for the 2 D USS Momentum Transfer Derivation
Leading to Laminar Velocity Profiles in a tube.

Step 3. Rate of momentum and Force Balance
Momentum rate in – Momentum rate out + sum of Forces = 0

$$2\pi L\left((r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}\right) + 2\pi r\Delta r\left(P_o - P_L\right) + 2\pi r\Delta r L pg = 0$$

(2.45)

Step 4. Divide by 2πLΔr and take limit to get a differential equation

$$\frac{(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}}{\Delta r} = -\frac{r\left(P_o - P_L\right) + rLpg}{L} \qquad (2.46)$$

In the limit as each delta term approaches 0

$$\frac{\partial\left(r\tau_{rz}\right)}{\partial r} = \left[pg + \frac{\left(P_o - P_L\right)}{L}\right]r \qquad (2.47)$$

Step 5. Substitute Newton's Law of Viscosity: $\tau_{rz} = -\eta\dfrac{\partial V_z}{\partial r}$

$$\frac{\partial\ r(\partial V_z/\partial r)}{\partial r} = -\frac{r}{\eta}\left[pg + \frac{\left(P_o - P_L\right)}{L}\right] \qquad (2.48)$$

Expanding the first term and dividing by r gives

$$\left[\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right] = -\frac{1}{\eta}\left[pg + \frac{\left(P_o - P_L\right)}{L}\right] \qquad (2.49)$$

2.11 – The Analytical Solution to the 1D SS Rate of Momentum in Cylindrical Coordinates

BC #1 r = 0 ; V_z = max or $\tau_{rz} = 0$
BC #2 r = R; $V_z = 0$

Rearrange Eq. (2.47) to get

$$\partial\left(r\tau_{rz}\right) = \left[pg + \frac{\left(P_o - P_L\right)}{L}\right]r\partial r \qquad (2.50)$$

Perform indefinite integration to get

$$\tau_{rz} = \left[pg + \frac{\left(P_o - P_L\right)}{L}\right]\frac{r}{2} + C_1 \qquad (2.51)$$

Evaluate C_1 with BC #1 to show it to be 0. Then substitute in Newton's Law of Viscosity to get

$$-\frac{\partial V_z}{\partial r} = \frac{1}{\eta}\left[pg + \frac{(P_o - P_L)}{L}\right]\frac{r}{2} \tag{2.52}$$

which may be integrated indefinitely to yield

$$V_z = -\left[pg + \frac{(P_o - P_L)}{L}\right]\frac{r^2}{2} + C_2 \tag{2.53}$$

Using BC #2 to solve for C_2 results in the final equation

$$V_z = \left[pg + \frac{(P_o - P_L)}{L}\right]\left(\frac{R^2}{4\eta}\right)\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{2.54}$$

The maximum velocity is

$$V_{z,MAX} = \left[pg + \frac{(P_o - P_L)}{L}\right]\left(\frac{R^2}{4\eta}\right) \tag{2.55}$$

at r = 0.

This result is known as the Hagen-Poiseuille Equation and describes laminar flow through a cylinder.