Cantilevered Beams

For a load on a cantilevered beam with modules E and moment of inertia I,

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 y}{dx^2} \right) = q \tag{1}$$

If E, I, and q are functions of distance along the beam, then Eq. (1) may be written

$$\frac{d^2}{dx^2}\left(f(x) \frac{d^2 y}{dx^2} \right) = q(x) \tag{2}$$

where

$$f(x) = E(x) I(x) \tag{3}$$

Expanding Eq. (2) gives

$$f''(x)\frac{d^2 y}{dx^2} + 2 f'(x)\frac{d^3 y}{dx^3} + f(x)\frac{d^4 y}{dx^4} = q(x) \tag{4}$$

Solving for the fourth derivative gives

$$\frac{d^4 y}{dx^4} = \left(q(x) - f''(x)\frac{d^2 y}{dx^2} - 2 f'(x)\frac{d^3 y}{dx^3} \right) \Big/ f(x) \tag{5}$$

For a Cantilevered beam, the moment and shear are zero at the free end (x=L). The shear Q and the moment M are given by

$$Q = f(x)\frac{d^3 y}{dx^3} \tag{6}$$

and

$$M = f(x)\frac{d^2 y}{dx^2} \tag{7}$$

The deflection and slope of the beam are the values y and the first derivative $y'(x)$.

Summarizing, Eq. (5) is the 4th derivative y'''' that when integrated

- one time is y'''= Q/f(x)
- two times is y'' = M/f(x)
- three times is y' = slope
- four times is y = deflection.

The four boundary conditions for the above integrations are as follows:

- Q=0 at x=L
- M=0 at x=L
- y'=0 at x=0
- y=0 at x=0

The initial value array “I” will be entered in an order consistent with the derivative array “D” as follows: I=[y_o, y'_o, y''_o, y'''_o] as and D(x,y)=[y’, y’’, y’’’, y’’’’] both as column arrays. The value of y’’’’ is known to be Load(x)/(E(x)I(x)). The problem is complicated by the unknown values for y''_o and y'''_o. The *sbval* function versomes this problem by iteratively solving for them. The argument call for the *sbval* function according to direct quote from MathCad is as follows:

> **sbval(v, x1, x2, D, load, score)** returns a set of initial conditions for the boundary value problem specified by the derivatives in D and guess values in v on the interval [x1,x2]. Parameter load contains both known initial conditions and guess values from v, and score measures solution discrepancy at x2.

Solutions

Solutions for a cantilevered beam are presented below for two cases:

a) Constant IE and load q
b) Variable I(x)E(x) and load q(x)

Constant IE and load q

The MathCad solution for the unknown boundary conditions at x=0 for the second and third derivatives is shown below.

Cantilevered Beam *Constant IE and load*

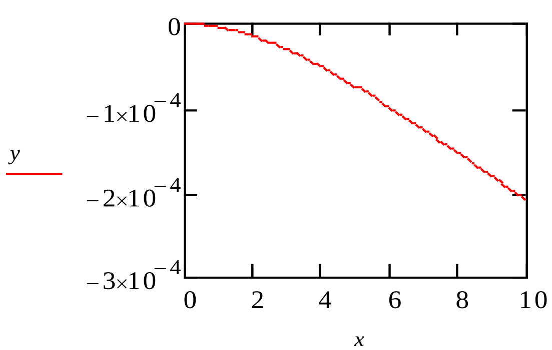
$I := 200$ $E := 3 \cdot 10^{7}$

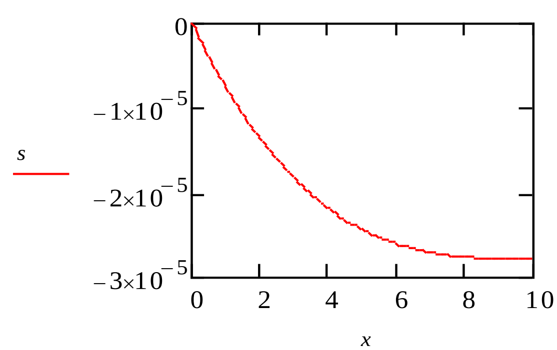
$f(x) := I \cdot E$ $f1(x) := 0$ $f2(x) := 0$ $q(x) := -1000$

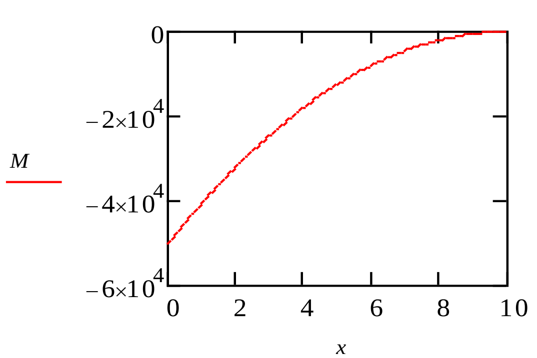
$$D(x,z) := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \dfrac{\left(q(x) - f2(x) \cdot z_2 - 2 \cdot f1(x) \cdot z_3\right)}{f(x)} \end{bmatrix} \quad load(x,w) := \begin{pmatrix} 0 \\ 0 \\ w_0 \\ w_1 \end{pmatrix}_0 \quad w := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad score(x,z) := \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$$

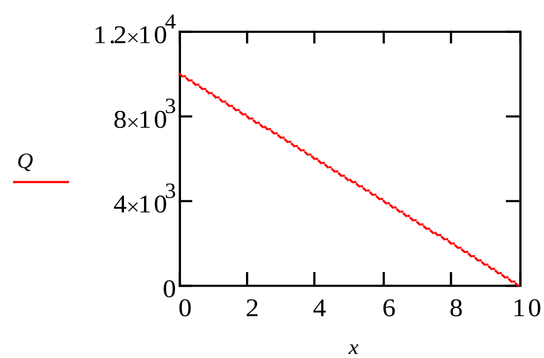
$$sol := sbval(w, 0, 10, D, load, score) \quad I := \begin{pmatrix} 0 \\ 0 \\ sol_0 \\ sol_1 \end{pmatrix} \quad Ans := rkfixed(I, 0, 10, 100, D)$$

$$x := Ans^{\langle 0 \rangle} \quad y := Ans^{\langle 1 \rangle} \quad s := Ans^{\langle 2 \rangle} \quad M := Ans^{\langle 3 \rangle} \cdot f(x) \quad Q := Ans^{\langle 4 \rangle} \cdot f(x)$$









Variable IE and load q

Cantilevered Beam *Variable IE and load*

$i_0 := 200$ $i_1 := -10$ $e_0 := 2\cdot 10^7$ $e_1 := 10^6$ $I(x) := i_0 + i_1\cdot x$ $E(x) := e_0 + e_1\cdot x$

$f(x) := I(x)\cdot E(x)$ $f1(x) := i_1\cdot e_0 + 2\cdot i_1\cdot e_1\cdot x$ $f2(x) := 2\cdot i_1\cdot e_1$ $q(x) := -200 - 100\cdot x$

$$D(x,z) := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \dfrac{\left(q(x) - f2(x)\cdot z_2 - 2\cdot f1(x)\cdot z_3\right)}{f(x)} \end{bmatrix} \quad load(x,w) := \begin{pmatrix} 0 \\ 0 \\ w_0 \\ w_1 \end{pmatrix} \quad w := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad score(x,z) := \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$$

$$sol := sbval(w, 0, 10, D, load, score) \quad I := \begin{pmatrix} 0 \\ 0 \\ sol_0 \\ sol_1 \end{pmatrix} \quad Ans := rkfixed(I, 0, 10, 100, D)$$

$$x := Ans^{\langle 0\rangle} \quad y := Ans^{\langle 1\rangle} \quad s := Ans^{\langle 2\rangle} \quad M := Ans^{\langle 3\rangle}\cdot f(x) \quad Q := Ans^{\langle 4\rangle}\cdot f(x)$$